BONFERRONI BOUND PROBLEM AND ITS RELATED POLYHEDRON

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By

ARUN KUMAR PUJARI

to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

DECEMBER, 1979

CERTIFICATE

This is to certify that the work embodied in the thesis 'BONFERRONI BOUND PROBLEM AND ITS RELATED POLYHEDRON' by Arun Kumar Pujari has been carried out under our supervision and has not been submitted elsewhere for a degree or, diploma.

(A.K. Mittal)

(S.K. Gupta)

Indian Institute of Technology
KANPUR

December, 1979

MATH-1979-D-PUJ-BON.

ACKNOWLEDGEMENT

I take this opportunity to record the deep sense of gratitude I feel towards my advisors Dr. Gupta and Dr. Mittal for their incessant guidance, suggestions and advice during the course of this work. But for their inspiration this thesis could not have taken its present shape. I am highly indebted and grateful to them.

I express my thanks to my colleague Mr. V.P. Gulati for his innovative suggestions and stimulating discussions many a time when I was bogged down by some difficulty or other.

I am grateful for all the help rendered by my friends and colleagues. They have been the silent pillars of strength on which I could lean on.

Mr. G.L. Misra and Mr. S.K. Tewari deserve my acknowledgement for their efficient typing.

— Arun —

CONTENTS

Chapter I

INTRODUCTION

1.1 *Linear Bonferroni Bound Probem*

The problem of estimating the probabilities of unions and intersections of events when all the information required about the event is not known, is one of the oldest unsolved problems in probability theory. Generally the probability of the objective event is to be estimated in the sense of giving its sharpest upper and lower bounds in terms of the available data. The linear Bonferroni Bound problem is a problem of this kind. The origin of such type of problem goes as far back as to Boole [7]. But it is only after the work by Frechet [11], the problem has acquired its present status. Frechet determines the sharpest upper and lower bounds for the probabilities of unions and intersections in terms of their respective probabilities

In view of the fact that Frechet's result cannot be used for estimating the probabilities of all compound events, several workers ([5], [8], [10], [13] - [17], [19], [20], [29] - [31], [33], [34], [37], [40], [41], [47], [52]), have been motivated to extend the result to wider classes of problems. Linear Bonferroni Bound problem is also the result of such extensions.

Let $A_1, A_2, \ldots, A_n$ be any sequence of events. Let $P_{i_1 i_2 \ldots i_\nu} = P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_\nu})$ for all $1 \leq i_1 < i_2 < \ldots < i_\nu \leq n$. Consider the following linear inequality,

$$\bar{x}_0 + \sum_{1 \leq i \leq n} \bar{x}_i P_i + \sum_{1 \leq i_1 \leq i_2 \leq n} \bar{x}_{i_1 i_2} P_{i_1 i_2} + \ldots$$

$$\ldots + \sum_{1 \leq i_1 < i_2 < \ldots i_\nu \leq n} \bar{x}_{i_1 i_2 \ldots i_\nu} P_{i_1 i_2 \ldots i_\nu}$$

$$\leq P(\bigcup_{i=1}^{n} A_i)$$

$$\leq \underline{x}_0 + \sum_{1 \leq i \leq n} \underline{x}_i P_i + \sum_{1 \leq i_1 < i_2 \leq n} \underline{x}_{i_1 i_2} P_{i_1 i_2} + \ldots$$

$$\ldots + \sum_{1 \leq i_1 < i_2 < \ldots < i_\nu \leq n} \underline{x}_{i_1 i_2 \ldots i_\nu} P_{i_1 i_2 \ldots i_\nu}. \qquad (1.1.1)$$

Then the linear Bonferroni bound problem of degree ν is to determine the real numbers $\bar{x}_0$, $\underline{x}_0$, $\bar{x}_i$, $\underline{x}_i, \ldots, \bar{x}_{i_1 i_2 \ldots i_\nu}$, $\underline{x}_{i_1 i_2 \ldots i_\nu}$, such that the inequality (1.1.1) provides the sharpest possible bounds of $P(\bigcup_{i=1}^{n} A_i)$. The problem of determining the sharpest possible bounds of $P(\bigcap_{i=1}^{n} A_i)$ in the similar manner is also called the linear Bonferroni bound problem.

Though a classical problem in probability theory, there is no satisfactory solution of the linear Bonferroni bound problem up to now. At present the exact computation of the bounds of degree two seems to be very difficult problem except

for few special cases, such as when $P_i = P_1$, $P_{ij} = P_{12}$, for all $1 \leq i < j \leq n$, the bounds are readily available. Thus searching for a sufficiently precise bound is still a challenging problem.

In the present work we restrict our scope to finding the Bonferroni lower bound of degree two for $P(\bigcup_{i=1}^{n} A_i)$. In the following section we give a brief account of the available lower bounds for this problem.

1.2 *Known Bounds*

The most commonly known lower bound is

$$P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P_i - \sum_{1 \leq i < j \leq n} P_{ij}. \qquad (1.2.1)$$

Chung and Erdos [8] improve the bound (1.2.1). Lemma 1.1 summarizes the result by Chung and Erdos. This lower bound is also reestablished by Whittle [52].

<u>Lemma 1.1</u> [8], [52]

$$P(\bigcup_{i=1}^{n} A_i) \geq \frac{s_1^2}{s_1+2s_2}, \qquad (1.2.2)$$

provided that $P(\bigcup_{i=1}^{n} A_i) > 0$, where

$$s_1 = \sum_{i=1}^{n} P_i \quad \text{and} \quad s_2 = \sum_{1 \leq i < j \leq n} P_{ij}.$$

The lower bound (1.2.2) is exact if $P_i = P_j = P_{ij}$, $\forall\ 1 \leq i < j \leq n$.

Lemma 1.2 gives the bound obtained by Dowson and Sankoff [10], who improve the result of lemma 1.2, by employing linear programming technique.

Lemma 1.2 [10]

$$P(\bigcup_{i=1}^{n} A_i) \geq \frac{2s_1}{(k+1)} - \frac{2s_2}{k(k+1)}, \qquad (1.2.3)$$

where (k-1) is the largest integer less than $\frac{2s_2}{s_1}$ and $P(\bigcup_{i=1}^{n} A_i) > 0$. This bound is exact if $n = 2$.

Galambos [15] shows that the lower bound (1.2.3) is the sharpest possible bound if the bound is to be expressed in terms of s_1 and s_2 instead of P_i, P_{ij}.

Gallot [17] and Kounias [29] develop another kind of bound using matrix analysis. The following lemma summarizes their result.

Lemma 1.3 [17], [29]

$$P(\bigcup_{i=1}^{n} A_i) \geq P^T . Q^- . P, \qquad (1.2.4)$$

where $P = (P_1, P_2, \ldots, P_n) \in R^n$ and Q^- is any generalised inverse of the matrix $Q = (P_{ij})$, satisfying $QQ^-Q = Q$. The matrix Q has the following elements

$$P_{ii} = P_i \quad \text{and } P_{ij} = P_{ji} \text{ for all } 1 \leq i < j \leq n.$$

In fact Gallot [17] obtains the above result for only nonsingular Q and Kounias [29] extends the result to singular cases.

Kounias also shows that the bound (1.2.4) is exact if Q has rank one, since then all events coincide, or if it has rank two i.e. there are essentially two events and one is a subset of the other i.e.

$$P_1 \geq P_2 = P_{12} \quad \text{or,} \quad P_2 \geq P_1 = P_{12} \text{ [29].}$$

It has also been shown by Kounias [29] that

$$P(\bigcup_{i=1}^{n} A_i) \geq \underset{J_r}{\text{Max}} \left[\sum_{i \in J_r} P_i - \sum_{\substack{i < j \\ i,j \in J_r}} P_{ij} \right], \tag{1.2.5}$$

where for $2 \leq r \leq n$, $J_r \subseteq \{1,2,\ldots,n\}$ having r elements.

Sobel and Uppuluri [47] give lower bounds of degree two for the cases when $P_i = P_1$, $P_{ij} = P_{12}$ for all $1 \leq i < j \leq n$. The lower bound obtained by them is

$$P(\bigcup_{i=1}^{n} A_i) \geq \max_{2 \leq r \leq n} \left[r\,P_1 - \frac{r(r-1)}{2} P_{12} \right] \tag{1.2.6}$$

The most recent lower bound of degree two is obtained by Kounias and Marin [31]. This bound provides an improvement over all other earlier bounds i.e. (1.2.1) - (1.2.6). Kounias and Marin formulate the problem as linear programming problem. Since the present work is primarily based on their work, we give details of the work in following sections of this chapter.

1.3 *Linear Programming Formulation* [31]

In this section the Bonferroni lower bound problem is formulated as a linear programming problem. The following

notations will be used throughout.

Let us consider the $\frac{n(n+1)}{2}$ -dimensional vector space $R^{\frac{n(n+1)}{2}}$. For notational convenience, we shall denote the components of any element X in this space as follows.

The components of any $X \in R^{\frac{n(n+1)}{2}}$ are denoted by x_i, $1 \leq i \leq n$, and x_{ij}, $1 \leq i < j \leq n$; and are arranged as shown below,

$$
\begin{array}{l}
x_1 \to x_2 \to x_3 \to x_4 \to \cdots \to x_n \\
\quad \to x_{12} \to x_{13} \to x_{14} \to \cdots \to x_{1n} \\
\qquad \to x_{23} \to x_{24} \to \cdots \to x_{2n} \\
\qquad\qquad \ddots \\
\qquad\qquad\qquad \to x_{n-1n}.
\end{array}
$$

Thus by $X \in R^{\frac{n(n+1)}{2}}$ we shall mean the vector

$$X = (x_1, x_2, \ldots, x_n, \; x_{12}, \ldots, x_{1n}, x_{23}, \ldots, x_{2n} \; \ldots \; x_{n-1n}).$$

To denote any vector in this space, it is only necessary to specify the elements x_i, $1 \leq i \leq n$, and x_{ij}, $1 \leq i < j \leq n$.

Let $\bar{P} \in R^{\frac{n(n+1)}{2}}$ be the vector with the corresponding $x_i = P_i$ and $x_{ij} = -P_{ij}$.

Similarly, for a given $J \subseteq N = \{1, 2, \ldots, n\}$, we define the vector $d(J) \in R^{\frac{n(n+1)}{2}}$ as follows.

$d(J)$ has components d_i, d_{ij}, $1 \leq i < j \leq n$ such that,

$$d_i = 1 \quad \text{if } i \in J,$$
$$\;\; = 0 \quad \text{if } i \notin J,$$

and $$d_{ij} = -d_i \cdot d_j, \quad \forall\, 1 \le i < j \le n.$$

Let C_o denote the convex hull of the set $\{d(J) \mid J \subseteq N\}$.

Let us consider the inequality

$$P\left(\bigcup_{i=1}^{n} A_i\right) \ge x_o + \sum_{i=1}^{n} x_i P_i - \sum_{1 \le i < j \le n} x_{ij} P_{ij}. \tag{1.3.1}$$

It is well known that, when $x_o = 0$ and $x_i = x_{ij} = 1, 1 \le i < j \le n$, the inequality (1.3.1), which reduces to

$$P\left(\bigcup_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P_i - \sum_{1 \le i < j \le n} P_{ij}, \tag{1.3.2}$$

holds for any arbitrary sequence of events A_i, $1 \le i \le n$. The sharpest lower bound of the type (1.3.1) can be obtained by determining the variables x_o, x_i, x_{ij}, $1 \le i < j \le n$ such that the right hand side of the inequality (1.3.1) takes on its maximum value.

Let E be any sequence of events $A_1, A_2, \ldots, A_n$ such that each event is either the whole probability space or, null. Thus for such a sequence E, $P_i = 0$ or, 1 and $P_{ij} = P_i\, P_j$. Clearly d(J) defines the vector $\bar{P}$ associated with the sequence E, when A_i, $i \in J$ are full space and A_i, $i \notin J$ are null. Such a sequence of events is called an elementary sequence. Obviously there are 2^n elementary sequences. Renyi [41] shows that

any linear inequality of the type (1.3.1) holds true for an arbitrary sequence of events iff it holds for all the 2^n elementary sequences. Kounias and Marin [31] reestablish the result by using method of random variables. Galambos [15], Hailperin [19] also derive a similar result by using linear programming technique. Since $\bar{P} \in C_0$, Renyi's result follows from the fact that if any linear inequality holds true for all the points d(J), then it also holds true for any arbitrary point of C_0.

Thus writing the inequality (1.3.1) for all the elementary sequences, we have

$$x_0 \leq 0, \tag{1.3.3}$$

$$x_0 + d(J)^T X = x_0 + \sum_{i \in J} x_i - \sum_{\substack{i < j \\ i,j \in J}} x_{ij} \leq 1, \tag{1.3.4}$$

for all $\phi \neq J \subseteq N$.

Thus the sharpest bound of the type (1.3.1) can be obtained by

Maximize

$$x_0 + \sum_{i=1}^{n} x_i P_i - \sum_{1 \leq i < j \leq n} x_{ij} P_{ij} \tag{1.3.5}$$

subject to the conditions (1.3.3) – (1.3.4).

Reformulation of linear program :

The above linear programming problem can be reformulated to a simpler form, with the purpose to make the problem a little easier. For this reformulation the following results are useful.

Lemma 1.4 [31]

If for any x_o, x_i, x_{ij}, $1 \leq i < j \leq n$, satisfying (1.3.3) - (1.3.4), x_o is strictly negative, then the point $\hat{x}_o = 0$, $\hat{x}_i = \frac{x_i}{1-x_o}$, $\hat{x}_{ij} = \frac{x_{ij}}{1-x_o}$ $1 \leq i < j \leq n$ also satisfies the set of constraints (1.3.3) - (1.3.4).

Proof : Since x_o, x_i, x_{ij}, $1 \leq i < j \leq n$ is a feasible solution to the set of constraints (1.3.3) - (1.3.4) we have

$$x_o + \sum_{i \in J_r} x_i - \sum_{\substack{i < j \\ i,j \in J_r}} x_{ij} \leq 1 \quad \text{for any } J_r \subseteq \{1,2,\ldots,n\}$$

$$\Longrightarrow \frac{1}{1-x_o} \{ \sum_{i \in J_r} x_i - \sum_{\substack{i < j \\ 1,j \in J_r}} x_{ij} \} \leq 1$$

$$\Longrightarrow \sum_{i \in J_r} \hat{x}_i - \sum_{\substack{i < j \\ i,j \in J_r}} \hat{x}_{ij} \leq 1.$$

Hence the point $\hat{x}_o = 0$, $\hat{x}_i = \frac{x_i}{1-x_o}$, $\hat{x}_{ij} = \frac{x_{ij}}{1-x_o}$ $1 \leq i < j \leq n$ satisfies all the constraints (1.3.3) - (1.3.4).

Thus for any solution of the system (1.3.5) – (1.3.4) such that $x_o < 0$, there corresponds another feasible solution $(\hat{x}_o, \hat{x}_i, \hat{x}_{ij})$ with $\hat{x}_o = 0$. The following lemma shows that feasible points of later type need to be considered for the optimization purposes.

Lemma 1.5 [31]

There exists atleast one optimal solution of the linear programming problem (1.3.3) – (1.3.5) such that $x_o = 0$ at this optimal solution.

Proof : Assume that (x_o^*, x_i^*, x_{ij}^*) $1 \leq i < j \leq n$, is the point at which (1.3.5) takes on its maximum value, subject to the restrictions (1.3.3) – (1.3.4) using the result of lemma 1.4, we construct a new feasible solution as

$$x_o = 0, \; x_i = \frac{x_i^*}{1-x_o^*}, \; x_{ij} = \frac{x_{ij}^*}{1-x_o^*} \quad 1 \leq i < j \leq n.$$

The value of (1.3.5) at the point $(0, x_i, x_{ij})$ is given by

$$\left\{ \sum_{i=1}^{n} P_i x_i^* - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij}^* \right\} \frac{1}{(1-x_o^*)} = \frac{z}{(1-x_o^*)} \quad (1.3.6)$$

where $z = \sum_{i=1}^{n} P_i x_i^* - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij}^*$.

It is well known that, from (1.3.5)

$$x_o^* + z \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq 1$$

$$\Longrightarrow -x_o^*(x_o^* + z - 1) \leq 0 \qquad \because x_o^* < 0$$

$$\Longrightarrow x_o^* - x_o^* \cdot z - x_o^{*2} \leq 0$$

$$\Longrightarrow z - x_o^{*2} - x_o^* z + x_o^* \leq z$$

$$\Longrightarrow (1-x_o^*)(x_o^* + z) \leq z$$

$$\Longrightarrow x_o^* + z \leq \frac{z}{1-x_o^*} \tag{1.3.7}$$

The left hand side of (1.3.7) is the value of the objective function (1.3.5) at the point (x_o^*, x_i^*, x_{ij}^*). Hence, if (x_o^*, x_i^*, x_{ij}^*) is the maximum point then, $x_o^* + z = \frac{z}{1-x_o^*}$.

Hence $(0, x_i, x_{ij})$ is also the optimal point.

This completes the proof.

Thus using lemma 1.4 and 1.5, the variable x_o can be dropped from the linear programming problem. Hence the resulting linear programming is :

To determine $X \in R^{\frac{n(n+1)}{2}}$, which

$$\text{LP}(N): \quad \text{Maximizes} \sum_{i=1}^{n} P_i x_i - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij} \tag{1.3.8}$$

$$\text{subject to} \sum_{i \in J} x_i - \sum_{\substack{i < j \\ i,j \in J}} x_{ij} \leq 1 \tag{1.3.9}$$

for all nonempty subsets J of N.

We will now onwards be concerned with the problem LP(N).

1.4 *Motivation*

Problem LP(N) is a linear program of $\frac{n(n+1)}{2}$, unrestricted variables subject to 2^n-1 constraints. As n increases, the number of constraints becomes unsurmountable, and hence none of the the conventional linear programming algorithms is capable of effectively handling such large size problems. However some of its special features are encouraging. It is observed that for fixed n, the set of constraints (1.3.9) remains independent of the sequence of events A_i, of their respective probabilities P_i, P_{ij} and so does the convex polyhedron defined by the set of constraints (1.3.9). Moreover, for different values of n the coefficient matrix of the set of constraints being similarly structured, the geometry of the convex polyhedron is similar and hence the set of extreme points of the polyhedron in $R^{\frac{n(n+1)}{2}}$ may follow a particular structure. This important feature of LP(N) is due to the special structure of the constraints and of the right hand side constants. Hence, notwithstanding the size of the problem, it may be possible to develop a special purpose algorithm which can effectively exploit the structure of LP(N). Motivated by this observation, an attempt is made to explore the structure of LP(N) and to develop an approximate algorithm based on its special structure.

1.5 *Basic Approach*

The approach followed in the present work is mainly influenced by the work of Kounias and Marin [31]. This section

gives a brief outline of their approach.

Let F(N) denote the convex polyhedron defined by the set of inequalities (1.3.9). J_k denotes the subset of N, having k elements where $N = \{1,2,\ldots,n\}$.

Definition 1.1

$X \in F(N) \subseteq R^{\frac{n(n+1)}{2}}$ is said to be an extreme point of F(N) if it satisfies $\frac{n(n+1)}{2}$ linearly independent constraints of (1.3.9) as equalities.

Obviously, the convex polyhedron F(N) is unbounded. But since the maximum value of (1.3.8) is bounded at 1, the maximum value will be finite and hence it is attained on atleast one extreme point of F(N). In a rather unconventional way, Kounias and Marin define a point $X \in F(N)$ to be a vertex point of F(N) if X satisfies $\frac{n(n+1)}{2}$ constraints as equalities and these constraints need not be linearly independent. Since any extreme point is also a vertex point, it can be directly concluded that atleast at one of the vertex points the objective function takes on its maximum value. Thus maximizing the objective function (1.3.8) over the set of all vertex points of F(N) we get the optimal value of LP(N).

The approach used by Kounias and Marin is to generate 8 classes of vertex points, V_1, $V_2,\ldots,V_8$, where the optimization can be carried out. Though, in the process, all the vertex points for $n \leq 4$ are obtained, these 8 classes do not cover all

the vertex points of $F(N)$ for larger values of n. However the maximum value of (1.3.8) over $\bigcup_{i=1}^{8} V_i$ gives a lower bound better than any of the previous bounds discussed in section 1.2.

These 8 classes of vertex points are as follows :

$$V_1 : x_i = \begin{cases} x_i^* , & i \in J_r, \\ 0 , & i \notin J_r \end{cases} \qquad x_{ij} = \begin{cases} x_{ij}^* , & i,j \in J_r,\ i < j, \\ 0 , & \text{otherwise}, \end{cases}$$

$3 \leq r < n$, where x_i^*, x_{ij}^* are the coordinates of any vertex point of $F(J_r)$; (1.5.1)

$$V_2 : x_i = \begin{cases} \frac{2}{k} , & i \in J_r, \\ 0 , & i \notin J_r, \end{cases} \qquad x_{ij} = \begin{cases} \frac{2}{k(k-1)} , & i,j \in J_r, \\ 0 , & \text{otherwise}, \end{cases}$$

for all $2 \leq k \leq r-1$ and $k = 2$, if $n = 2$; (1.5.2)

$$V_3 : x_i = \begin{cases} \frac{2}{k} , & i \in J_{n-1}, \\ -\frac{2}{k-1} , & i \in J_1, \end{cases} \qquad x_{ij} = \begin{cases} \frac{2}{k(k-1)} , & i,j \in J_{n-1}, \\ -\frac{2}{k(k-1)}, & i \in J_1, j \in J_{n-1}, \end{cases}$$

where $J_1 \cap J_{n-1} = \phi$, for all $2 \leq k \leq n-2$; (1.5.3)

$$V_4 : x_i = \begin{cases} \frac{2}{k} , & i \in J_{n-1}, \\ \frac{1}{k} , & i \in J_1, \end{cases} \qquad x_{ij} = \begin{cases} \frac{2}{k(k-1)} , & i,j \in J_{n-1}, \\ \frac{1}{k(k-1)} , & i \in J_1,\ j \in J_{n-1}, \end{cases}$$

$J_1 \cap J_{n-1} = \phi$, for all $3 \leq k \leq n-2$; (1.5.4)

$$V_5 : x_i = \begin{cases} \frac{2}{k}, & i \in J_{n-1}, \\ \frac{1}{k}, & i \in J_1, \end{cases} \qquad x_{ij} = \begin{cases} \frac{2}{k(k-1)}, & i,j \in J_{n-1}, \\ -\frac{1}{k(k-1)}, & i \in J_1, j \in J_{n-1}, \end{cases}$$

$J_1 \cap J_{n-1} = \phi$, for all $2 \leq k \leq n-3$; (1.5.5)

$$V_6 : x_i = \begin{cases} \frac{2}{k}, & i \in J_{n-2}, \\ -\frac{1}{k-1}, & i \in J_2, \end{cases} \qquad x_{ij} = \begin{cases} \frac{2}{k(k-1)}, & i,j \in J_{n-2}, \\ -\frac{1}{k(k-1)}, & i \in J_2,\ j \in J_{n-2}, \\ 0, & i,j \in J_2, \end{cases}$$

for $J_2 \cap J_{n-2} = \phi$, for $2 \leq k \leq n-3$; (1.5.6)

$$V_7 : x_i = \begin{cases} \frac{2}{k}, & i \in J_{n-2}, \\ \frac{1}{k}, & i \in J_2, \end{cases} \qquad x_{ij} = \begin{cases} \frac{2}{k(k-1)}, & i,\ j \in J_{n-2}, \\ \frac{1}{k(k-1)}, & i \in J_2,\ j \in J_{n-2}, \\ 0 & i,j \in J_2, \end{cases}$$

$J_2 \cap J_{n-2} = \phi$, for all $2 \leq k \leq n-2$; (1.5.7)

$V_8 : x_i = 2b_i - b_i^2$, $x_{ij} = 2b_i b_j$, $1 \leq i < j \leq n$, (1.5.8)

where $\sum_{i \in J} b_i = 1$ for atleast $\frac{n(n+1)}{2}$ subjects J of N.

The following results are due to Kounias and Marin [31].

Let L_i be the maximum value of (1.3.8) over $X \in V_i$, for $1 \leq i \leq 8$.

<u>Theorem 1.1</u>

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \max_{1 \leq i \leq 8} L_i \qquad (1.5.9)$$

Theorem 1.2

$$\max_{1 \leq i \leq 8} L_i$$ is sharper lower bound than the bounds (1.2.1) - (1.2.6).

Thus Kounias and Marin give the best known lower bound by maximizing (1.3.8) over the 8 classes of vertex points. But it is not the best possible bound always. Moreover for the maximization purposes it suffices to concentrate only on the set of extreme points, instead of the set of vertex points which is larger than the former set. Keeping this in mind it can be said that a successful characterization of the set of all extreme points together with a tractable algorithm to optimize the objective function over the set, would give a better way of computing the best lower Bonferroni bound than any of the conventional approaches. A partial characterization of the set of extreme points is given in this thesis.

Following the same approach of Kounias and Marin, a broader class of feasible points is defined. The maximum value of the objective function over this class improves the bound obtained by Kounias and Marin. Since the trivial way of getting the best bound is to solve the linear program by simplex method, though at a high cost, emphasis is given on numerical computation of best lower bound at a low cost. For several special cases some simple ways of computing the exact lower bound are discussed.

1.6 *Thesis Summary*

Besides the current chapter, the thesis contains 4 more chapters.

In Chapter II we introduce a new class of feasible points V^* of F(N). It is shown that the maximum value of (1.3.8) over this class provides the present best lower bound. Since, no efficient technique to compute such improved bounds could be developed, we give two different computationally tractable algorithms to compute two different types of lower bounds. It is also shown that one of these two bounds is always better than previously known bounds. Numerical computation of improved bounds involves minimization of a quadratic convex function of 0-1 variables. A branch and bound technique is proposed to solve such problems. Besides finding lower bound, attempt is made in this chapter to solve linear program LP(N) through its dual by the use of column generation technique. It is shown that subproblems arising from column generation technique are equivalent to minimization of a quadratic Pseudo-Boolean function. Computational aspects of such problems are not explored to its full extent.

In Chapter III, we deal with the structure of the extreme points of the convex polyhedron F(N). Two different kinds of extreme points are characterised. The first kind is characterised in terms of the coordinates of the points. For the second kind of extreme points, though the explicit characterisation of the

coordinates is not given, it is shown that such points can be obtained from a type of feasible points whose coordinates are known. Besides this, a partial characterisation of all the extreme points of F(N) is given. Every attempt to generate all the extreme points of F(N) proved abortive. However, some developments in this direction are reported and a conjecture is stated leading to generation of all the extreme points. The adjacency structure of special kind of extreme points is also discussed in this chapter.

In Chapter IV, some particular cases are studied for which the best Bonferroni bound can be computed without much effort.

In Chapter V, we explore the possibility of generalizing the methods studied so far, for the cases of Bonferroni bounds of higher degree. We show that the class V^* can be generalised for such cases. The structure of some types of extreme points of the corresponding polyhedron is also given.

Chapter II

BEST LOWER BOUND

2.1 *Introduction*

The problem of best Bonferroni lower bound is formulated in section 1.3 as the linear program :

$$\text{LP}(N): \quad \begin{aligned} &\text{To maximize} \\ &z = \sum_{i=1}^{n} x_i P_i - \sum_{1 \leq i < j \leq n} x_{ij} P_{ij}, && (2.1.1) \\ &\text{subject to} \sum_{i \in J} x_i - \sum_{\substack{i < j \\ i,j \in J}} x_{ij} \leq 1 && (2.1.2) \end{aligned}$$

for all nonempty subsets $J \subseteq N = \{1,2,\ldots,n\}$.

Considering the difficulties encountered in solving LP(N), attempt is made to maximize the linear function (2.1.1) over a smaller set of feasible points. The maximum value of (2.1.1) over any subset of the feasible region provides a lower bound which may not necessarily be the sharpest bound. A desirable characterization of such a subset of F(N) is that it should be small enough to be easily handled for maximization purposes and large enough to give a considerably stringent bound. In this chapter a class of feasible points, denoted by V^*, is defined and it is shown that the maximum value of (2.1.1) over this class is sharpest among the known bounds (1.2.1) - (1.2.6) and (1.5.9). The maximization problem over the class leads to

maximizing a sequence of unconstrained quadratic functions of n integer variables. It is to be noted that, though the LP(N) is of $\frac{n(n+1)}{2}$ variables, a reduction of the number of variables is obtained at the expense of the linearity of the objective function. Similarly, it would be seen that by imposing integer restrictions on the variables, the set of constraints (2.1.2) are automatically taken care of. Though no exact method to solve such quadratic-integer optimization problem is discussed, an approximate algorithm to get a lower bound is developed by defining a type of local maxima. Such approximate method leads to maximizing a Pseudo-Boolean quadratic concave function. A branch and bound technique is developed to solve such a problem. We report our computational result in section 2.8. In section 2.9 we show that in order to get the best known lower bound one needs to maximize a concave Pseudo-Boolean quadratic function of 2n variables. Such problems can be solved by the proposed branch and bound method. In the last section we attempt to solve the dual of LP(N) by column generation technique and show that the subproblems are equivalent to minimizing Pseudo-Boolean quadratic functions.

2.2 *The Class V^**

In this section a class of feasible points is defined and some properties of this class are studied.

Let the class $V^* \subseteq R^{\frac{n(n+1)}{2}}$ be defined as follows :

For any $X \in V^*$

$$\left.\begin{aligned} x_i &= \frac{(2h+1)\,a_i - a_i^2}{h(h+1)}\,, && 1 \le i \le n \\ x_{ij} &= \frac{2a_i\,a_j}{h(h+1)}\,, && 1 \le i < j \le n \end{aligned}\right] \qquad (2.2.1)$$

where a_i, $1 \le i \le n$, and $h \neq -1, 0$ are arbitrary integers.

For any given $(n+1)$-dimensional integer vector $(a,h) = (a_1, a_2, \ldots, a_n, h)$, using (2.2.1) a point $X \in V^*$ can be obtained. There can be several such integer vectors (a,h) which define the same point $X \in V^*$. For instance,

let $a_1 = h$, $a_2 = h+1$, $a_i = 0$, $i > 2$, h arbitrary.

Then the corresponding X is

$$x_1 = 1,\; x_2 = 1,\; x_{12} = 2,\; x_i = x_{ij} = 0, \text{ for } 2 < i < j \le n. \qquad (2.2.2)$$

Clearly the point (2.2.2) is generated for any arbitrary h and hence there are infinitely many (a,h) which define the same point X of V^*. In theorem 2.1, we show that we can restrict h to be a positive integer and still get all the points of V^*.

Theorem 2.1

Any $X \in V^*$ obtained by (a,h) can also be obtained by $(-a,-h-1)$ using the relation (2.2.1).

Proof : From (2.2.1), we have,

$$x_i = \frac{(2h+1)\,a_i - a_i^2}{h(h+1)}\,, \quad x_{ij} = \frac{2a_i a_j}{h(h+1)}\,, \quad 1 \le i < j \le n.$$

The point generated by the vector $(-a,-h-1)$ is given by

$$\frac{\{2(-h-1)+1\}\{-a_i\}-\{-a_i\}^2}{(-h-1)\,(-h)} = \frac{(2h+1)\,a_i - a_i^2}{h(h+1)} = x_i$$

$$\frac{2(-a_i)\,(-a_j)}{(-h-1)(-h)} = \frac{2a_ia_j}{h(h+1)} = x_{ij}.$$

This completes the proof.

Thus, in order to define all the points of V^* it suffices to consider the (n+1)-dimensional integer vectors (a, h) with h positive. Although there is no one-one correspondence between such set of integers and the points of V^*, only for the sake of simplicity we shall notationally use $(a,h) \in V^*$, $h \geq 1$, as the point $X \in V^*$ obtained from (a, h) by (2.2.1).

Let V_R^* denote the class of points $X \in R^{\frac{n(n+1)}{2}}$ obtained from (a, h), $h \geq 1$, by the formula (2.2.1) where the integer restriction on a is relaxed. Similarly, we define V_B^* as the subset of V^* where in relation (2.2.1) a_i's are restricted to the values -1, 0, 1 and 2, and $h \geq 1$, integer. For a fixed h, define a subset $V^*(h)$ of V^* as;

$$V^*(h) = \{(a,h) \mid a \text{ is an integer vector}\}.$$

The subsets $V_R^*(h)$, $V_B^*(h)$ can be similarly defined. Let L_R^*, L^*, L_B^*, $L_R^*(h)$, $L^*(h)$, $L_B^*(h)$ be the maximum values of (2.1.1) over the classes V_R^*, V^*, V_B^*, $V_R^*(h)$, $V^*(h)$, $V_B^*(h)$, respectively.

The following theorem shows that V^* is a subset of the feasible region F(N).

Theorem 2.2

$$V^* \subseteq F(N)$$

Proof : Let us consider the constraint (2.1.2) corresponding to any arbitrary subset $J \subseteq N$.

For any $X = (a,h) \in V^*$, substituting the values of x_i, x_{ij} from (2.2.1), we have,

$$\sum_{i \in J} x_i - \sum_{\substack{i < j \\ i,j \in J}} x_{ij}$$

$$= \sum_{i \in J} \frac{(2h+1)\, a_i - a_i^2}{h(h+1)} - \sum_{\substack{i < j \\ i,j \in J}} \frac{2a_i\, a_j}{h(h+1)}$$

$$= \frac{1}{h(h+1)} \left[(2h+1) \sum_{i \in J} a_i - \left(\sum_{i \in J} a_i \right)^2 \right]$$

$$= \frac{1}{h(h+1)} \left[h - \sum_{i \in J} a_i \right] \left[\sum_{i \in J} a_i - h - 1 \right] + 1 \tag{2.2.2}$$

For integers a_i and $h \neq -1, 0$, the following inequality holds,

$$\frac{1}{h(h+1)} \{ h - \sum_{i \in J} a_i \} \{ \sum_{i \in J} a_i - h - 1 \} + 1 \leq 1 \tag{2.2.3}$$

hence, from (2.2.2) we have,

$$\sum_{i \in J} x_i - \sum_{\substack{i < j \\ i,j \in J}} x_{ij} \leq 1$$

Since the inequality (2.2.3) holds for any $J \subseteq N$, hence the point $X = (a,h) \in V^*$ satisfies all the constraints (2.1.2).

Hence $V^* \subseteq F(N)$ and this completes the proof.

In the theorem 2.3, it is shown that the class V^* is a more general class than any of V_i's defined in [31]. Let $C[V^*]$ denote the convex hull of V^*.

<u>Theorem 2.3</u>

$$V_i \subseteq C[V^*] \quad \text{for } 2 \leq i \leq 7.$$

<u>Proof</u> : We prove the theorem in parts.

(i) In this part it is shown that $V_2 \subset V^*$, $V_3 \subset V^*$. Putting $a_i = 1$, $i \in J$ and $a_i = 0$ $i \notin J$ in (2.2.1) for any $J \subseteq N$, we get

$$x_i = \frac{2}{h+1} \quad i \in J, \quad x_{ij} = \frac{2}{h(h+1)} \quad i < j,\ i,j \in J,$$

$x_i = x_{ij} = 0$ otherwise.

From (1.5.2) clearly for $h = k-1$, $2 \leq k \leq |J| - 1$, and $k = 2$, if $n = 2$, the above point is in V_2. Substituting $a_i = 1$, $i \in J_{n-1}$, $a_i = -1$, $i \in J_1$ in (2.2.1) we get

$$x_i = \frac{2}{h+1}, \quad x_{ij} = \frac{2}{h(h+1)} \quad i \lessgtr j,\ i,j \in J_{n-1}$$

$$x_i = -\frac{2}{h}, \quad x_{ij} = -\frac{2}{h(h+1)} \quad i \in J_1,\ j \in J_{n-1}.$$

From (1.5.3) clearly for $h = k-1$, $3 \leq h \leq n-1$, $J_1 \cap J_{n-1} = \phi$, the above point is in V_3.

(ii) In this part, it is shown that points in V_i, $4 \leq i \leq 7$ can be expressed as the convex combinations of points in V^*.

Let us consider the pair of points (a',h) and (a'',h) where,

$a'_i = 1,\ i \in J_n$

$a''_i = 1,\ i \in J_{n-1},\ a''_i = 0,\ i \in J_1,$ for $J_1 \cap J_{n-1} = \phi$.

Then the point $X \in V_4$, from (1.5.4), is given by

$$X = \frac{1}{2} X' + \frac{1}{2} X'' \text{ where } X' = (a',h) \in V^* \text{ and}$$

$$X'' = (a'',h) \in V^*.$$

To see this, from (2.2.1) we have,

$$x'_i = \frac{2}{k},\ x'_{ij} = \frac{2}{k(k-1)} \quad i < j,\ i,j \in J_n$$

$$x''_i = \frac{2}{k},\ x''_{ij} = \frac{2}{k(k-1)} \quad i < j,\ i,j \in J_{n-1}$$

and $x''_i = 0 = x''_{ij}$ for $i \in J_1,\ j \in J_{n-1}$.

Thus $$x_i = \frac{1}{2} x'_i + \frac{1}{2} x''_i = \begin{cases} \frac{2}{k},\ i \in J_{n-1} \\ \frac{1}{k},\ i \in J_1, \end{cases}$$

$$x_{ij} = \frac{1}{2} x'_{ij} + \frac{1}{2} x''_{ij} = \begin{cases} \frac{2}{k(k-1)},\ i < j,\ i,j \in J_{n-1}, \\ \frac{1}{k(k-1)},\ i \in J_1,\ j \in J_{n-1}. \end{cases}$$

Thus from (1.5.4), for $3 \leq k \leq n-2$, the point X is in V_4.

Similarly, any point $X \in V_5$ can be expressed by the points X', $X'' \in V^*$ defined as follows;

$X' = (a',h) \in V^*$, such that $a'_i = 1,\ i \in J_{n-1},\ a'_i = 0,\ i \in J_1$

$X'' = (a'',h) \in V^*$, such that $a''_i = 1,\ i \in J_{n-1},\ a''_i = -1,\ i \in J_1$

where J_1, J_{n-1} correspond to subsets defined in (1.5.5).
It is merely a routine to check that for $2 \leq k \leq n-3$ any $X \in V_5$ can be written as,

$$X = \frac{1}{2} X' + \frac{1}{2} X''.$$

Similarly the following pair of points give rise to the point in V_6 as their convex combination. For any $X \in V_6$, define

$$X' = (a', h) \in V^* \text{ such that}$$

$$a_i' = 1,\ i \in J_{n-2},\ a_i = 0,\ i \in J_1' ,\ a_i = 1,\ i \in J_1$$

and $X'' = (a'', h) \in V^*$ such that

$$a_i'' = 1,\ i \in J_{n-2},\ a_i'' = 1,\ i \in J_1',\ a_i'' = 0,\ i \in J_1$$

where $J_2 = J_1 \cup J_1'$ and J_{n-2} are defined in (1.5.6).
Thus for $h = k-1$, $2 \leq k \leq n-2$, any $X \in V_6$ can be written as $X = \frac{1}{2} X' + \frac{1}{2} X''$.

Similarly for the case of $X \in V_7$, i.e. from (1.5.7)

$$x_i = \frac{2}{k+1} ,\ x_{ij} = \frac{2}{k(k+1)} ,\ i \in J_{n-2},$$

$$x_i = \frac{1}{k+1} ,\ i \in J_2,\ x_{ij} = \frac{1}{k(k+1)} ,\ i \in J_2,\ j \in J_{n-2}, x_{ij} = 0, \text{otherwise}$$

we define the pair of points $X' = (a', h) \in V^*$, $X'' = (a'', h) \in V^*$ such that,

$$a_i' = 1,\ i \in J_{n-2},\ a_i' = 1,\ i \in J_1,\ a_i' = 0,\ i \in J_1'$$

$$a_i'' = 1,\ i \in J_{n-2},\ a_i'' = 0,\ i \in J_1,\ a_i'' = 1,\ i \in J_1'$$

Then, clearly $X = \frac{1}{2} X' + \frac{1}{2} X''$.

Thus part (i) and part (ii) complete the proof.

Corollary 2.1 :

$$V_i \subseteq C [V_B^*] \quad \text{for } 2 \leq i \leq 7.$$

Proof : It is to be noted that in the proof of theorem 2.3, in order to show that $V_i \subseteq C[V^*]$ for $2 \leq i \leq 7$, the values required for any a_i are 0, 1, -1 only. Thus for any $X \in V_i$, $2 \leq i \leq 7$ there exist $X', X'' \in V_B^*$ such that $X \in C[X', X'']$.

2.3 *Best lower bound*

Clearly $V^* \subseteq F(N)$

$$\Longrightarrow L^* \leq \text{Max of (2.1.1) subject to (2.1.2)}$$

$$\Longrightarrow L^* \leq P(\bigcup_{i=1}^{n} A_i).$$

Thus L^* is a lower bound of $P(\bigcup_{i=1}^{n} A_i)$. In this section, it is shown that L^* improves the bounds obtained by Kounias and Marin [31] and hence from theorem 1.2, L^* is the best known bound.

Theorem 2.4

$$L^* \geq \max_{1 \leq i \leq 8} L_i.$$

Proof : In view of theorem 2.3 we need only to prove that

$$L^* \geq \text{Max}(L_1, L_8).$$

The proof is in two parts, the first part shows $L^* \geq L_8$ and the second part, $L^* \geq L_1$.

(i) Substituting the value of x_i, x_{ij} from (1.5.8) the objective function is,

$$\Sigma P_i(2b_i - b_i^2) - \sum_{1 \leq i < j \leq n} 2b_i\, b_j\, P_{ij}$$

$$= 2\,\bar{P}^T b - b^T Q b \quad \text{where } b = (b_1, \ldots, b_n) \in R^n \qquad (2.3.2)$$

The function (2.3.2) takes its maximum at the point which satisfies

$$Qb = P$$

or, $\quad b = Q^- P$ for a generalised inverse Q^- of Q [29] (2.3.3)

The maximum value of (2.3.2) at (2.3.3) is given by

$$2P^T Q^- P - b^T Q^T Q^- Q b = 2P^T Q^- P - P Q^- P = P Q^- P \qquad (2.3.4)$$

Since V_8 is defined by b_i, $1 \leq i \leq n$ from (1.5.8) where b_i's are subjected to restriction that at least for $\frac{n(n+1)}{2}$ constraints $\Sigma\, b_i = 1$, hence clearly

$$L_8 \leq P^T Q^- P$$

Since any irrational number can be approximated to a rational number to any degree, it may be assumed that b_i's are rational numbers. Thus,

$$b_i = \frac{p_i}{q_i} \quad \text{for all } i, \qquad (2.3.5)$$

where p_i, q_i are relatively prime numbers and $q_i > 0$. Let $h = \text{LCM}\,[q_1, q_2, \ldots, q_n]$. Then

$$b_i = \frac{p_i}{q_i} = \frac{a_i}{h}, \qquad (2.3.6)$$

where a_i is integer, for all $1 \leq i \leq n$.

Let us consider the point $(a,h) \in V^*$, where a_i, $1 \leq i \leq n$ and h are defined in (2.3.6).

Substituting the value of $(a,h) \in V^*$ in (2.1.1), we have,

$$\sum_{i=1}^{n} \frac{(2h+1)\, a_i - a_i^2}{h(h+1)} P_i - \sum_{1 \leq i < j \leq n} \frac{2\, a_i a_j}{h(h+1)} P_{ij}$$

$$= \{(2h+1)\, P^T a - a^T Q a\} \; \frac{1}{h(h+1)} \tag{2.3.7}$$

Substituting $\frac{1}{h}\, a = b$ from (2.3.6) in (2.3.7)

$$= \{(2h+1)\, P^T b - h\, b^T Q b\} \; \frac{1}{(h+1)}$$

$$= \{(2h+1)\, P^T Q^- P - h\, P^T Q^- P\} \; \frac{1}{h+1}$$

$$= P^T Q^- P.$$

(ii) In this part we show that $L^* \geq L_1$.

Like $X \in R^{\frac{n(n+1)}{2}}$, let X^r, a point in $R^{\frac{r(r+1)}{2}}$, be defined as $X^r = (x_1^r, x_2^r, \ldots, x_r^r, x_{12}^r, \ldots, x_{1r}^r, \ldots, x_{r-1r}^r)$ $X(0,r)$ denotes the point in $R^{\frac{n(n+1)}{2}}$ obtained from X^r as follows;

$$x_i(0,r) = x_i^r,\; x_{ij}(0,r) = x_{ij}^r \text{ for } i < j,\; i,j \in J_r \subseteq N.$$

$$x_i(0,r) = x_{ij}(0,r) = 0 \quad \text{for } i \notin J_r.$$

Let V_r^* be the set of points of the kind V^* in $R^{\frac{r(r+1)}{2}}$ i.e., $V_r^* \subseteq F(J_r)$ is defined by $r+1$ integers a_i, $i \in J_r$, $h \geq 1$ from

the relation (2.2.1). Similarly we define $(V_i)_r$ as the vertex points of the kind V_i in $F(J_r)$. Clearly,

$$X^r \in V_r^* \Longrightarrow X(0,r) \in V^*,$$

and the value of the objective function at X^r, and $X(0,r)$ are same.

Similarly, $X^r \in (V_i)_r \Longrightarrow X(0,r) \in V_i \quad \forall\ i = 1,2,\ldots,8.$
The class V_1, from (1.5.1), is the class of points of the type $X(0,r)$, where X^r is a vertex point of $F(J_r)$. Since there are only 8 classes of vertex points known so far, we shall use the following definition of V_1 for all practical purposes

$$V_1 = \{ X(0,r) \mid X^r \in \bigcup_{i=2}^{h} (V_i)_r \text{ for all } 1 \leq r \leq n \}.$$

Hence

$$L_1 = \underset{1 \leq r \leq n-1, J_r}{\text{Max}} \quad \underset{X^r \in \bigcup_{i=2}^{8} (V_i)_r}{\text{Max}} \left\{ \sum_{i \in J_r} P_i x_i^r - \sum_{\substack{i < j \\ i,j \in J_r}} P_{ij} x_{ij}^r \right.$$

$$\leq \underset{1 \leq r \leq n-1, J_r}{\text{Max}} \quad \underset{X^r \in (V^*)_r}{\text{Max}} \left\{ \sum_{i \in J_r} P_i x_i^r - \sum_{\substack{i < j \\ i,j \in J_r}} P_{ij} x_{ij}^r \right.$$

$$= \underset{1 \leq r \leq n-1, J_r}{\text{Max}} \quad \underset{X(0,r) \in V^*}{\text{Max}} \left\{ \sum_{i=1}^{n} P_i x_i(0,r) - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij}(0,r) \right.$$

$\leq L^*$

Thus $L_1 \leq L^*$.

This completes the proof.

Corollary 2.2

$$L_B^* \geq L_i, \qquad 2 \leq i \leq 7.$$

Proof : Proof directly follows from corollary 2.1.

$$C[V_B^*] \supseteq V_i^*, \quad 2 \leq i \leq 7$$

$$\implies L_B^* \geq L_i, \qquad 2 \leq i \leq 7.$$

2.4 *Numerical computation of lower bound*

In this section computational aspect of the bound L^* is discussed. It is shown that the problem of computing L^* leads to hard combinatorial optimization problem.

Substituting the value of x_i, x_{ij} from (2.2.1) in (2.1.1) we have,

$$\text{Max } z1 = \frac{(2h+1)\, a^T P - a^T Q a}{h(h+1)} \tag{2.4.1}$$

P1 :

subject to $a \in Z^n$, $h \geq 1$ integer, where Z is the set of integers.

Lemma 2.1

For a fixed $h > 0$, z1 is a concave function of a for all $a \in R^n$.

Proof : For fixed h,

$$z1 = \frac{1}{h(h+1)} \left[(2h+1)\, a^T P - a^T Q a \right]$$

is obviously a quadratic function of $a \in R^n$. Since Q is a covariance matrix [29], it is positive semi-definite. So z1

is a concave function of $a \in R^n$. Hence the problem of computing $L^*(h)$ reduces to maximizing a concave quadratic function of n integer variables. The apparent simplicity of the problem is deceptive. We fail to develop any effective algorithm to solve such problems. For some unknown reasons such problems are constantly ignored even in literature. Some relevant references are [1] - [3], [6], [21] - [28], [32], [35], [36], [38], [42] - [45], [53]. Basically two types of problems have been dealt with so far. First, the maximization of any unconstrained quadratic function of integer variables [6], [9], [21] - [28], [42] - [44]. Second, the maximization of concave quadratic function of integer variables subject to linear constraints and non-negative restrictions on the variables [1] - [3], [32], [35], [36], [38]. It is found that none of these algorithms are suitable to take due advantages of the structure of the problem under consideration. The algorithms of the first kind are incapable of making use of the concavity of the function and so they are not recommended for the present problem. The second type of algorithms are heavily dependent on the set of constraints, that is, on the boundedness of the feasible region. Hence these algorithms are not efficient in solving any unconstrained maximization problem.

Thus it is difficult to develop an effective algorithm to find $L^*(h)$. Moreover, $L^*(h)$ which depends on h, cannot be easily expressed as a simple algebraic function of h. Unless the property of $L^*(h)$ as a function of h is known, even in the

presence of effective algorithm to maximize unconstrained, concave quadratic function of integer variables, it may be difficult to determine

$$L^* = \max_{\substack{h \geq 1 \\ h \text{ integer}}} L^*(h).$$

In view of these difficulties an approximate method to solve P1 is given here. The approximation of L^* in the proposed algorithm is made with respect to the following conditions.

(1) Instead of searching for a global maxima of concave quadratic function with integer variables, a kind of local maximizing point is searched. This local optima is obtained by rounding off the global maximizing points when the variables are not restricted to be integers. For the problem under consideration such approximation seems to be good. For all the problems encountered during computation such local maximizing points were also observed to be global.

(2) In order to compute $L^* = \max_{h \geq 1} L^*(h)$, h is restricted to be less than n. This assumption is based on the computational experience. Thus there are only n quadratic maximization problems to be solved.

With the help of these assumptions an algorithm is developed which finds an approximate value of L^* in a finite number of steps.

The locally maximizing point is defined as follows;

Definition 2.1 (Locally maximizing points)

Let a^* be the real maximum point obtained by relaxing the integer restriction. Let the components a_i^* of a^*, $i \in J$, be fractions and for $i \notin J$ be integers, where $|J| = m$ where $1 \leq m \leq n$. A locally maximizing point is the best point among the set of 2^m integers obtained from a^*, by fixing the component a_i^*, $i \notin J$ and by rounding-up or, -down the components a_i^*, $i \in J$.

We concentrate on finding such locally maximizing points of P1. This problem, as we show below, boils down to a 0-1 optimization problem.

For a fixed h, a^* can be determined by method of differentiation as,

$$a^* = \frac{2h+1}{2} \; Q^{-}P \tag{2.4.2}$$

Thus substituting the value a^* from (2.4.2) we have,

$$L^*(h) = \frac{(2h+1)\; a^*P - a^{*T}Qa^*}{h(h+1)}$$

$$= \frac{2a^{*T}Qa^* - a^{*T}Qa^*}{h(h+1)}$$

$$= \frac{(2h+1)^2}{4h(h+1)}\; P^TQ^{-}P. \tag{2.4.3}$$

Without loss of generality, we can assume that all the components of a^* are fractional and let δ_i be the fractional part of a_i^*. So we have

$$a_i^* = [a_i^*] + \delta_i \tag{2.4.4}$$

where $[a_i^*]$ is the largest integer less than a_i^*. Let δ be the vector $(\delta_1, \delta_2, \ldots, \delta_n) \in R^n$. For such a a^* and hence for a given δ we define a set of vectors

$$W = \{ a^* - \delta + y \mid y \in B_2^n \} \tag{2.4.5}$$

where B_2^n denotes the set of n-dimensional vector with components 0 or, 1.

Now the locally maximizing point is obtained by solving

$$\underset{c \in W}{\text{Max}} \quad (2h+1)\, c^T P - c^T Q c \tag{2.4.6}$$

Substituting the value of c for (2.4.5) in (2.4.6) we have

$$(2h+1)\, (a^* - \delta + y)^T P - (a^* - \delta + y)^T Q (a^* - \delta + y)$$

$$= \Big[(2h+1)\, (a^{*T}P - \delta^T P + y^T P) - a^{*T} Q a^* - \delta^T Q \delta - y^T Q y$$

$$+ 2a^{*T} Q \delta + 2\delta^T Q y - 2a^{*T} Q y \Big]$$

$$= \Big[(2h+1) a^{*T} P - a^{*T} Q a^* - \delta^T Q \delta - y^T Q y + 2\delta^T Q y$$

$$- (2h+1)\delta^T P + (2h+1)\, P^T y + 2a^{*T} Q \delta - 2a^{*T} Q y \Big]$$

(using relation (2.4.3))

$$= (2h+1) a^{*T} P - a^{*T} Q a^* - \delta^T Q \delta - y^T Q y + 2\delta^T Q y$$

$$= M - y^T Q y + 2\delta^T Q y \tag{2.4.7}$$

where $M = (2h+1)\, a^{*T}P - a^{*T}Qa^* - \delta^T Q\delta$.

Thus the problem (2.4.7) is equivalent to

$$P2 : \quad \underset{y \in B_2^n}{\mathrm{Min}} \; y^T Qy - 2\delta^T Qy \qquad (2.4.8)$$

Let $z_I(h)$ denote the optimal value of P2.

2.5 *Algorithm 1 : to solve P 1*

In this section we describe the proposed algorithm to get an approximate solution to the problem P1.

Algorithm 1 : To solve P1.

Step one : Initialisation; $h = 1,\ u = n$

Step two : Find $a^* = \frac{2h+1}{2} Q^{-}P$, $L_R^*(h) = \frac{(2h+1)^2}{4h(h+1)} P^T Q^{-} P$.

Step three: Solve the problem P2 for the $a^* \in R^n$ derived in step two. The value $z_I(h)$ is computed.

Step four : The current Bonferroni lower bound is computed as

$$\hat{L} = (2h+1)\, a^{*T}P - a^{*T}Qa^* - \delta^T Q\delta - z_I(h).$$

Step five : Compute the integer k such that

$$L_R^*(k) \leq \hat{L} \leq L_R^*(k+1).$$

Step six : Update $u = \mathrm{Min}\,(k,u)$

$h = h+1$.

Step seven: If $h > u$, stop, otherwise, go to step two.

The efficiency and finiteness of the algorithm depend on those of solving the problem P2. But for the step three, the algorithm is simple and has at most n steps.

Thus the whole attempt to get an improved lower bound boils down to solving the problem P2. In the following section we shall discuss the technique to solve P2.

2.6 Branch and bound algorithm:for general 0-1 problem

The problem P2 can be viewed as minimization of Pseudo-Boolean function which is convex and quadratic. The importance of 0-1 program has led to the elaboration of numerous techniques for their solutions. Techniques which deal with optimization of quadratic convex polynomial are mainly branch and bound or, Boolean method and are discussed in [21] - [28], [42] - [44], [45], [53]. All these approaches are iterative, each iteration consisting of two phases, one of analysis of the given problem such as fixing the variables, finding good bounds for optima, and the other of synthesis of the new subproblem by branching or, back tracking. Boolean method is found to be more useful for analysis rather than for synthesis. Boolean analysis is not helpful for the problem P2. The proposed algorithm uses analysis similar to that used in works by Hammer [21] - [27] except that it does not make use of Boolean analysis.On the other hand, it follows the usual branch and bound technique suitably modified for the structure of the problem P2.

We give here, a brief outline of the branch and bound method to solve a general 0-1 optimization problem and develop the same with reference to P2 in section 2.7. Notation for branch and bound is followed as given in Murty [39].

Branch and bound method provides a methodology to search for an optimum feasible solution by enumerating partially a set of solutions and implicitly eliminating large groups of potential solutions to the problem without explicitly evaluating them. In the course of applying the branch and bound approach, the overall set of feasible solutions is partitioned into smaller subsets. At each stage one subset is chosen and an effort is made to find the best solution in it. If the best solution is found, then the subset is said to be fathomed. If it is not fathomed, then the subset is again partitioned into two subsets and the same process is repeated. The problems associated with the unfathomed subsets are called candidate problems.

With reference to a 0-1 optimization problem, the candidate problems are obtained by selecting a subset of variables and fixing them at 0 or, 1. Any variable that is fixed at 1 is called 1-variable and any variable that is fixed at 0 is called 0-variable. The 0-variables at 0 and 1-variables at 1, constitute a partial solution. Each candidate problem generated in the algorithm corresponds to a partial solution. Variables that are not included in the partial solution are called free variables in the candidate problem. Given a partial solution, a completion of it is obtained by giving values 0 or, 1 to each of the free variables.

We will first discuss the analysis to be performed on a typical candidate problem. Consider the candidate problem in which U_0, U_1, U_f are the sets of subscripts of the 0-, 1- and

free-variables respectively. For the given candidate problem, we shall determine tests, TEST1, TEST2, TEST3 so that TEST1 gives the fathoming criterion, TEST2 helps to augment the candidate problem to obtain a new candidate problem, and TEST3 partitions the set of potential solutions to two different subsets and defines two different candidate problems. If the current candidate problem is fathomed, a new candidate problem is picked up from the list of unfathomed candidate problems. Here, the candidate problem which is the latest entry in the list is taken as the current candidate problem. If the current candidate problem is augmented to a new candidate problem then the new candidate problem is taken as the current problem. If two candidate problems are generated from the current candidate problem then one of these is taken as the current problem and other is entered into the list of unfathomed problems.

At any stage, incumbent is the feasible solution that has the least objective value among the feasible solutions obtained from so far fathomed candidate problems. Let LB be the lower bound of the value of the objective function with respect to the current candidate problem. If LB is greater than the value of the present incumbent then the candidate problem is deleted from the list and this operation is called pruning. For pruning purpose, the lower bounding strategy which we adopt here, is as follows; all the integer restrictions of the free variables are relaxed and the minimum value of the objective function for the relaxed problem gives the lower bound, LB.

2.7 Algorithm 2: to solve P2

We now proceed to discuss the main features of the present algorithm with reference to P2.

The general form of P2 is

$$\text{Minimize} \quad -\sum_{i=1}^{n} c_i y_i + \sum_{i=1}^{n} P_i y_i^2 + 2 \sum_{1 \leq i < j \leq n} P_{ij}\, y_i\, y_j \tag{2.7.1}$$

for $y_i = 0,\ 1$ for $i \in N$, where $c_i = 2(\delta Q)_i$, defined in (2.4.8).

The typical candidate problem can be written as;

$$- \sum_{i \in U_1} c_i - \sum_{i \in U_f} \left[c_i - 2 \sum_{j \in U_1} P_{ij} \right] y_i + \sum_{i \in U_1} P_i + \sum_{i \in U_f} P_i\, y_i^2 + 2 \sum_{\substack{i < j \\ i,j \in U_f}} P_{ij}\, y_i\, y_j + 2 \sum_{\substack{i < j \\ i,j \in U_1}} P_{ij} \tag{2.7.2}$$

$$= H(U_1) - \sum_{i \in U_f} c_i(U_1)\, y_i + \sum_{i \in U_f} P_i y_i^2 + 2 \sum_{\substack{i < j \\ i,j \in U_f}} P_{ij}\, y_i\, y_j \tag{2.7.3}$$

where

$$H(U_1) = - \sum_{i \in U_1} c_i + \sum_{i \in U_1} P_i + 2 \sum_{\substack{i < j \\ i,j \in U_1}} P_{ij} \tag{2.7.4}$$

and

$$c_i(U_1) = c_i - 2 \sum_{j \in U_1} P_{ij} \tag{2.7.5}$$

Thus the objective function of the current candidate problem is given by (2.7.3). We next describe the method to determine TEST1, TEST2, TEST3.

In view of the fact that $y_i = y_i^2$ for $y_i = 0$ or, 1, the minimum value of (2.7.3) is same as,

$$\text{Min } H(U_1) + \sum_{i \in U_f} \left[P_i - c_i(U_1) \right] y_i + 2 \sum_{\substack{i < j \\ i,j \in U_f}} P_{ij}\, y_i\, y_j \tag{2.7.6}$$

for $y_i = 0$ or, 1 $\forall\, i \in U_f$.

The following results are utilized to develop the tests.

Lemma 2.2

If for some $k \in U_f$,

$$\left[-c_k(U_1) + P_k \right] \geq 0,$$

then $y_k = 0$ at the optimal solution of the problem (2.7.6).

Proof : The objective function can be rewritten as

$$H(U_1) + y_k \left[P_k - c_k(U_1) + 2 \sum_{j \in U_f \setminus \{k\}} P_{jk}\, y_j \right]$$

$$+ \sum_{j \in U_f \{k\}} \left[P_j - c_j(U_1) \right] y_j + 2 \sum_{\substack{i < j \\ i,j \in U_f \setminus \{k\}}} P_{ij}\, y_i y_j \tag{2.7.7}$$

Thus,

$$P_k - c_k(U_1) > 0$$

$$\Longrightarrow -c_k(U_1+P_k + 2 \sum_{j \in U_f \setminus \{k\}} P_{jk} y_j > 0 \quad \forall\, y_j = 0, \text{ or, } 1, \quad j \in U_f \setminus \{k\}$$

Thus the value of (2.7.6) with $y_k = 0$ is smaller than its value with $y_k = 1$, irrespective of the values of y_j, $j \neq k$.

Lemma 2.3

If for some $k \in U_f$,

$$-c_k(U_1) + P_k + 2 \sum_{j \in U_f \setminus \{k\}} P_{jk} < 0;$$

then $y_k = 1$ at the optimal solution of the problem (2.7.6).

Proof : From (2.7.7), it is clear that

$$- c_k(U_1) + P_k + 2 \sum_{j \in U_f \setminus \{k\}} P_{jk} < 0$$

$$\Longrightarrow -c_k(U_1) + P_k + 2 \sum_{j \in U_f \setminus \{k\}} P_{jk}\, y_j < 0$$

for all $y_j = 0$ or, 1, $j \in U_f \setminus \{k\}$.

$\Longrightarrow$ $y_k = 1$ at the optimal solution.

Thus for the candidate problem under consideration, by use of lemma 2.2, 2.3, some more variables can be fixed at the value 0 or, 1. Let U_0', U_1', U_f' be the set of subscripts of 0-, 1-, and free variables respectively, after successive use of lemma 2.2 and 2.3.

TEST1 : If $U_f' = \phi$, then the candidate problem is fathomed and the optimal value of the present candidate problem is $H(U_1')$.

TEST2 : If $U_f' \neq \phi$, the augmented candidate problem is

$$\text{Min } H(U_1') - \sum_{i \in U_f'} c_i(U_1')y_i + \sum_{i \in U_f'} P_i y_i^2 + 2 \sum_{\substack{i < j \\ i,j \in U_f'}} P_{ij}\, y_i\, y_j \tag{2.7.8}$$

subject to $y_i = 0$ or, 1, $\forall\ i \in U_f'$.

Since the problem (2.7.3) and (2.7.8) are of the same form, we shall consider (2.7.3) as the current candidate problem even for the case of (2.7.8).

At this stage, trivially,

$$-c_i(U_1) + P_i < 0, \quad \forall\ i \in U_f$$

and $\quad -c_i(U_1) + P_i + 2 \sum_{j \in U_f \setminus \{i\}} P_{ij} > 0, \ \forall\ i \in U_f,$

because, otherwise we can apply lemma 2.2, 2.3 successively, till we reach such a stage or till we fathom the candidate problem.

TEST3 : Select the subscript $m \in U_f$ such that

$$c_m(U_1) = \min_{j \in U_f} c_j(U_1).$$

We shall use y_m as the branching variable, and two subproblems are generated. The candidate problem 1 has the subscripts sets

as U_o, $U_1 \cup \{m\}$, $U_f \setminus \{m\}$, and the candidate problem 2 has subscripts sets as $U_o \cup \{m\}$, U_1, $U_f \setminus \{m\}$.

For the pruning purpose, the lower bound can be determined as given below. The problem (2.7.3) can be written in matrix form as,

$$D(U_1) - c(U_1)^T y + y^T Q(U_1) y \qquad (2.7.9)$$

where,

$c(U_1)$ is the vector $(c_i(U_1))$ of appropriate size,

y is the vector of (y_i) of appropriate size, and the matrix $Q(U_1)$ is obtained by eliminating i^{th} row, and i^{th} column of Q, for all $i \in U_1$.

Clearly, $Q(U_1)$, being a principal submatrix of Q, is positive semidefinite.

The unconstrained minimum value of (2.7.9) can be obtained by the method of differentiation, and is given by

$$L(U_1)$$

$$= - \frac{1}{2} \left[c(U_1)^T Q(U_1)^- c(U_1) \right] - H(U_1) \qquad (2.7.10)$$

Thus $L(U_1)$ gives the lower bound for the optimal solution of candidate problem (2.7.3).

The proposed algorithm works as follows;

Initially,

$$w = 0$$

$$U_o = \phi$$

$$U_1 = \phi .$$

If the current candidate problem is fathomed then, set $w = \text{Min}\ (H(U_1), w)$, where $H(U_1)$ gives the value of the present incumbent and a new candidate problem is picked up from the list. If the list is empty the algorithm stops. The subproblem which is listed last in the list is chosen as the current candidate problem. Otherwise, by application of TEST2 and TEST3, the current candidate problem is partitioned into two subproblems. The subproblem 1 is taken as the candidate problem and subproblem 2 is listed as unfathomed problems. If LB is the lower bound of the objective function for the candidate problem, then the branch leading from the problem is said to be pruned when $w \leq LB$. If it is not pruned, then algorithm is repeated with the current candidate problem, till the list becomes empty. At this stage the value of w gives the optimal value of the original problem.

Substituting $z_I(h) = w$, $\hat{L}$ is calculated.

<u>Remark</u> : Since the proposed branch and bound algorithm is aimed to be used in Algorithm 1, the initial value of w can be updated for different values of h from the information available about the $z_I(h-1)$. Such improved values of w help to reduce the total number of nodes to be scanned in Algorithm 2. The method of modification of initial values of w is given below.

(i) If $h = 1$, $w = (2h+1)\, a^{*T}P - a^{*T}Qa^{*} - \delta^{T}Q\delta - P^{T}Q^{-}P$

(ii) If $h > 1$, $w = (2h+1)a^{*T}P - a^{*T}Qa^{*} - \delta^{T}Q\delta - z_I(h-1)$

where a^{*} is defined in (2.4.2).

Though in general the branch and bound algorithm assumes $w = 0$ as its initial value, but while solving for Bonferroni bound problem such modification is very much useful.

Illustration

Following numerical example illustrates Algorithm 1 and Algorithm 2 for $n = 6$ and

$$Q = \begin{bmatrix} 0.62827 & 0.49564 & 0.20446 & 0.31045 & 0.32903 & 0.09525 \\ 0.49564 & 0.51940 & 0.22442 & 0.23636 & 0.23507 & 0.07645 \\ 0.20446 & 0.22442 & 0.29661 & 0.08655 & 0.11250 & 0.08274 \\ 0.31045 & 0.23636 & 0.03655 & 0.51358 & 0.35008 & 0.08274 \\ 0.32903 & 0.23507 & 0.11250 & 0.35008 & 0.57283 & 0.00000 \\ 0.09525 & 0.07645 & 0.08274 & 0.08274 & 0.00000 & 0.12142 \end{bmatrix}$$

Algorithm 1

$h = 1,\ u = n$

$a^* = (0.51135,\ 0.21314,\ 0.47680,\ 0.44461,\ 0.75346,\ 0.33674)$

$L_R^*(h) = L_R^*(1) = \frac{9}{8}\ P^TQ^-P = \frac{9}{8}\ (0.84948) = 0.95567$

$z_I(h) = z_I(1) = -1.80425$, obtained by using Algorithm 2.

$\hat{L} = 0.90213$

$k = 1$. Hence algorithm terminates here.

The best Bonferroni Bound is 0.90213 at

$$a_1 = 0,\ a_2 = 1,\ a_3 = 0,\ a_4 = 0,\ a_5 = 1,\ a_6 = 1,\ h = 1.$$

Algorithm 2

Initial lower bound is -1.91133

Initial upper bound $= w = -1.69896$.

$$U_1 = \{1,5\},\ U_o = \phi,\ U_f = N \setminus \{1,5\}$$

which has the value LB = -1.84714, and the new entry to list is the candidate problem

$$U_1 = \{1\},\ U_o = \{5\},\ U_f = N \setminus \{1,5\}$$

which has the LB values as -1.77521.

Using TEST1, TEST2 successively, the current candidate problem is fathomed. The complete solution is

$$y_1 = 1,\ y_2 = 0,\ y_3 = 0,\ y_4 = 0,\ y_5 = 1,\ y_6 = 1$$

and the value of the objective function at this point is -1.79647.

So the value of w is augmented to

$$w = -1.79647.$$

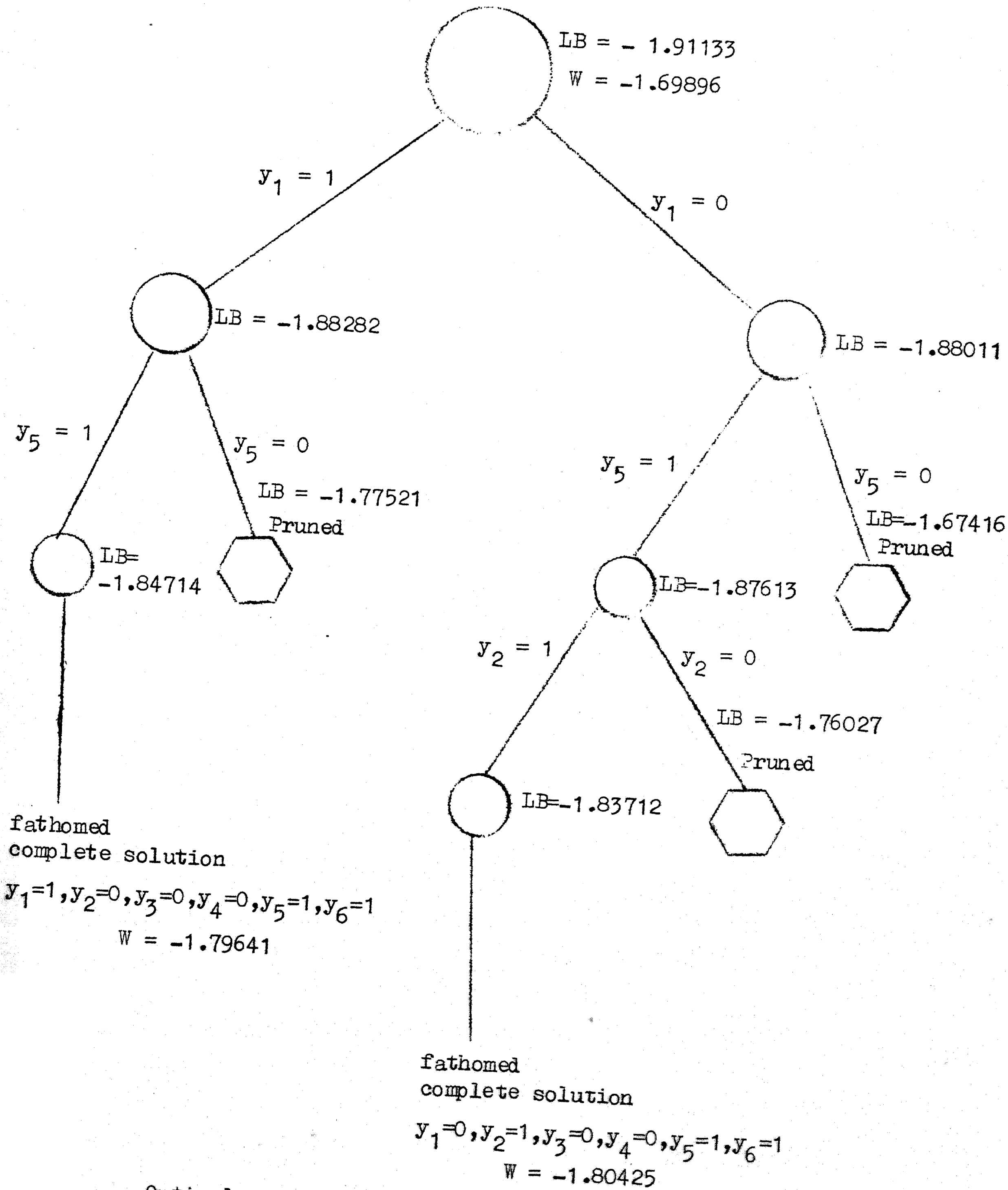
With this new value of w the list is pruned and latest candidate problem is deleted from the list.

Thus the algorithm is repeated till the list is empty. The whole algorithm is illustrated in fig. 2.1.

2.8 Computational results

In this section computational experience for Algorithm 1 and Algorithm 2 is reported. The algorithms are coded in FORTRAN-4 and run on DEC-10 (KL40 Processor) system. Algorithms are tested on randomly generated problems.

First we report the computational results for the branch and bound algorithm. In table 2.1, the results refer to average



Optimal value = -1.80425.

Time takenby DEC System 10 = 0.08 sec. (C.P.U.)

Fig. 2.1 : <u>BRANCH AND BOUND</u>

time and average number of nodes scanned, for randomly generated problems of different sizes as indicated.

Value of n	No. of problems solved	No. of nodes encountered	Time taken (C.P.U) in secs.
6	50	7	0.06
7	50	8	0.08
8	50	8	0.09
9	50	12	0.12
10	30	15	0.16
20	30	40	0.48
30	20	50	1.32
40	20	77	5.54
50	20	400	30
100	10	1263	135

Table 2.1

Next we discuss the computational experience for Algorithm 1. It is observed that for all the problems generated randomly, the approximate value of L^* obtained by Algorithm 1, is better than $\max_{1 \leq i \leq 8} L_i$ and thus for such problems Algorithm 1 gives the best lower bound.

In table 2.2 we give some typical problems of different dimensions. The average number of iterations and the average time taken by the algorithm is also given against the respective values of n.

Value of n	No. of problems solved	Average number of steps	Time taken (C.P.U.) in secs.	FOR A TYPICAL PROBLEM	
				Approx. L^*	$\underset{1 \leq i \leq 8}{\text{Max}} L_i$
6	20	3	0.16	0.94145	0.92753
7	20	3	0.20	0.88546	0.84317
8	20	3	0.31	0.85065	0.82429
9	20	3	0.40	0.89357	0.89342
10	10	5	1.01	0.95941	0.95509
20	5	7	3.57	0.93544	0.90859
30	2	15	90	0.93904	0.89953
40	2	20	130	0.90009	0.90009
50	2	20	180	0.88608	0.88608

Table 2.2

In table 2.3, we give for different problems for $n = 6$, the values L^* and $\underset{1 \leq i \leq 8}{\text{Max}} L_i$. The corresponding Q matrix is given in Appendix I.

Problem number	Approx. L^* by Algorithm 1	$\underset{1 \leq i \leq 8}{\text{Max}} L_i$
1	0.976	0.962
2	0.555	0.54833
3	0.72	0.695
4	0.88466	0.85147

Table 2.3

From the above results two important conclusions can be drawn;

(i) Previously there was no systematic method to compute a lower bound for high values of n. The present algorithm provides an efficient method to compute the bound for reasonable values of n.

(ii) Looking at the values of the improved bounds and at the number of steps taken by Algorithm 1, it is believed that we are justified by approximating L^*.

2.9 Computation of lower bound L_B^*

It is seen from corollary 2.2 that Max (L_B^*, L_8) also gives a bound better than the lower bounds obtained in [31]. In this section it is shown that computation of such best known lower bound leads to n maximization problems of concave quadratic Pseudo-Boolean function of 2n variables.

Substituting (a,h) for X, L_B^* can be determined as;

$$L_B^* = \underset{X \in V_B^*}{\text{Max}} \sum_{i=1}^{n} P_i x_i - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij}$$

$$= \underset{(a,h) \in V_B^*}{\text{Max}} \frac{(2h+1)\, a^T P - a^T Q a}{h(h+1)} \qquad (2.9.1)$$

As in (1.5.1) - (1.5.7), the value of k is restricted to $1 \leq k \leq n$, from corollary 2.1 and 2.2, it is clear that we need only to find the quantity

$$\underset{1 \leq h \leq n}{\text{Max}} L_B^*(h)$$

which improves the bounds L_i, $2 \leq i \leq 7$.

Putting

$$a_i = 2y_i + z_i - 1, \quad 1 \leq i \leq n \tag{2.9.2}$$

where, $y_i = 0$, or, 1, $z_i = 0$, or, 1, the problem of determing $L_B^*(h)$ reduces to

$$\underset{y \in B_2^n,\ z \in B_2^n}{\text{Maximize}} \quad \frac{1}{h(h+1)} \left[(2h+1)(2y+z-e)^T P - (2y+z-e)^T Q(2y+z-e) \right].$$

This 0-1 maximization problem is equivalent to

$$\text{P3 :} \quad \underset{\alpha \in B_2^{2n}}{\text{Minimize}} \quad \frac{1}{h(h+1)} \left[e^T Qe + (2h+1)e^T P - f^T \alpha + \alpha^T M\alpha \right]$$

where, e is a vector having components 1,

$$f^T = \left[(2(2h+1)P + 2e^T Q)^T, ((2h+1)P + e^T Q)^T \right] \in R^{2n},$$

$$\alpha = \begin{bmatrix} y \\ z \end{bmatrix} \in B_2^{2n},$$

$$M = \begin{bmatrix} 4Q & 2Q \\ 2Q & Q \end{bmatrix}.$$

It is easy to see that,

Q is positive semidefinite ==> M is positive semidefinite. Hence the objective function of P3 is convex. Moreover, M has nonnegative off-diagonal elements.

Hence the problem P3 can also be solved by Algoritm 2.

2.10 *Some properties of* V_R^*

It is observed from (2.2.2) that the integer restriction on a_i's and on h ensures feasibility of the set of points (a,h). Once this restriction is relaxed, the relation (2.2.1) gives rise to the class V_R^*. Unlike the case of V^*, any $X \in V_R^*$ need not automatically imply that $X \in F(N)$. For instance,

$$n = 2,\ h = 1,\ a_1 = 1 + \epsilon_1,\ a_2 = 1 + \epsilon_2,$$

where ϵ_1, ϵ_2 are in open interval (0,1), the point $(a,h) \in V_R^*$ does not satisfy all the constraints (2.2.2). On the contrary, if ϵ_1, ϵ_2 lie in the close-open interval $[1,\infty)$, the associated point $(a,h) \in V_R^*$ is feasible.

Thus clearly,

$$V^*(h) \subseteq V_R^*(h) \cap F(N)$$

$$\Longrightarrow L^*(h) \leq \max_{X \in V_R^*(h) \cap F(N)} \left[\sum_{i=1}^{n} P_i x_i - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij} \right],$$

h is fixed. (2.10.1)

$$\Longrightarrow L^*(h) \leq \max_{(a,h) \in V_R^*(h) \cap F(N)} \left[\frac{(2h+1)a^T P - a^T Q a}{h(h+1)} \right],$$

h is fixed

$$\leq L_R^*(h) \qquad (2.10.2)$$

Thus $$L^* \leq \max_{h \geq 1} L_R^*(h) \qquad (2.10.3)$$

Lemma 2.4 gives the value of $\max_{h \geq 1} L_R^*(h)$.

Lemma 2.4

$$\underset{h \geq 1}{\text{Max}} \; L_R^*(h) = \frac{9}{8} P^T Q^- P .$$

Proof : By the method of differentiation, we have;

$$L_R^*(h) = \frac{(2h+1)^2}{4h(h+1)} \; P^T Q^- P,$$

and it is attained at $(a = \frac{2h+1}{2} Q^- P, h)$.

Since Q is positive semidefinite, so is Q^-. Hence $P^T Q^- P \geq 0$.

Again $\frac{(2h+1)^2}{4h(h+1)}$ being a decreasing function of h, for $h > 0$,

$\frac{(2h+1)^2}{4h(h+1)} P^T Q^- P$ is also a decreasing function of h.

Hence $\underset{h \geq 1}{\text{Max}} \; L_R^*(h) = L_R^*(1) = \frac{9}{8} P^T Q^- P$.

This completes the proof.

Thus from (2.10.3),

$$L^* \leq \frac{9}{8} P^T Q^- P \tag{2.10.4}$$

The following observations are apparent.

(i) If $\frac{3}{2} Q^- P$ is an integer vector then

$$L^* = \frac{9}{8} P^T Q^- P .$$

This is so because the point $(\frac{3}{2} Q^- P, 1) \in V^*$ where $L_R^*(h)$ takes its maximum value $\frac{9}{8} P^T Q^- P$. Hence, from (2.10.4) we have $L^* = \frac{9}{8} P^T Q^- P$.

(ii) If for $a = \frac{3}{2} Q^- P$, $h = 1$, $(a,h) \in V_R^*$ is a feasible point of F(N), then the best known lower bound is $\frac{9}{8} P^T Q^- P$.

This is directly implied by (2.10.1), (2.10.3).

Thus lemma 2.4 gives a lower bound of L^*. Theorem 2.7 is useful in finding an upper bound of L^*. Besides this, it gives an insight to the geometrical structure of V_R^*.

Theorem 2.7

For any fixed $b \in R^n$, the set of points

$$\{(\frac{2h+1}{2}\, b, h) \mid h > 0\} \subseteq V_R^*$$

is a line in $R^{\frac{n(n+1)}{2}}$.

Proof : We prove the theorem by showing that the point $(\frac{2h'+1}{2}\, b, h') \in V_R^*$ lies on the line joining the origin and $(\frac{2h'-1}{2}\, b, h, h'-1) \in V_R^*$. Thus the set of points

$$\{(\frac{2h+1}{2}\, b, h) \mid h > 0\} \subseteq V_R^*,$$

gives rise to ray emanating from the origin.

$$\text{Let } X' = (\frac{2h'+1}{2}\, b, h'),$$

$$X'' = (\frac{2h''+1}{2}\, b, h''),\ h'' = h' - 1.$$

We claim that,

$$X' = \lambda X'',$$

where $\lambda = \dfrac{(2h'+1)^2(h'-1)}{(2h'-1)^2(h'+1)}$.

To see this,

$$x_i'' = \left[\frac{(2h'-1)(2h'-1)}{2} b_i - \frac{(2h'-1)^2}{4} b_i^2\right] \frac{1}{h'(h'-1)}$$

$$= \frac{(2h'-1)^2}{2} \left[b_i - \frac{b_i^2}{2}\right] \frac{1}{h'(h'-1)}$$

$$\lambda x_i'' = \frac{(2h'+1)^2}{2h'(h'+1)} (b_i - \frac{b_i^2}{2}) = x_i'$$

Similarly, $\lambda x_{ij}'' = x_{ij}'$.

It is not difficult to check that

$$0 \leq \lambda \leq 1.$$

Thus for any arbitrary h',

X' lines on the line joining X'' and the origin.

This completes the proof.

Corollary 2.4

For $h' > h''$, $C[V_R^*(h')] \subseteq C[V_R^*(h'')]$.

Proof : Since $0 \in V_R^*(h)$, $\forall h$, any point $(a,h') \in V_R^*(h')$ is the convex combination of

$$(\frac{2h''+1}{2h'+1} a, h'') \in V_R^*(h'') \text{ and the origin.}$$

Corollary 2.5

For any $h \geq 1$, if $(\frac{2h+1}{2} Q^-P, h) \in F(N)$ then

$$(\frac{2h'+1}{2} Q^-P, h') \in F(N), \quad \forall h' \geq h.$$

Proof : Since $F(N)$ is a convex set and the origin is also in $F(N)$, by using the result of theorem 2.7, the result follows.

From the above results it can be said that given an arbitrary vector $a \in R^n$, there always exists a $\bar{h}(a)$ such that for any $h \geq \bar{h}(a)$, the point $(\frac{2h+1}{2} a, h) \in F(N)$ and for any $h < \bar{h}(a)$, the point $(\frac{(2h+1)}{2} a, h) \notin F(N)$.

Let $\bar{h}$ denote the value $Q^{-}P(\bar{h})$. Then clearly $L_R^*(\bar{h})$ provides a lower bound.

We fail to use the above results as a tool for easy solution procedure. Nevertheless this promises a lower bound $L_R^*(\bar{h})$. It is believed that a feasible direction search technique can possibly be developed making use of the above results.

2.11 *Column generation techniqe to solve the dual*

In this section it is shown that the use of column generation technique to solve dual of LP(N) leads to maximizing a sequence of Pseudo-Boolean quadratic functions. But unlike the cases of P2 and P3, such problems can not be solved by Algorithm 2 discussed in 2.6, because the method assumes the function to be concave with positive off-diagonal elements of the associated quadratic form, which may not be true in the case of the subproblem generated by column generation.

Writing the primal linear program LP(N) in the matrix form, we have

$$LP(N) : \quad \begin{array}{l} \text{Max } \bar{P}^T X \\ \\ \text{subject } DX \leq e \end{array} \Bigg] \tag{2.11.1}$$

where e is a vector with all components 1, D is the coefficient matrix whose rows are $d(J)$, $\forall\ \phi \neq J \leq N$. The associated dual linear program $DP(N)$ is

$$DP(N) : \quad \begin{array}{l} \text{Minimize } e^T Y \\ \\ \text{such that } \quad D^T Y = \bar{P} \end{array} \tag{2.11.2}$$

$$Y \geq 0 \tag{2.11.3}$$

Since corresponding to every subset $J \subseteq N$, there is a component of the dual variable vector Y, we denote the components of Y as $y(J)$, $\forall\ J \subseteq N$.

The column generation technique starts with a known basic feasible solution Y_B of $DP(N)$. Let B be the associated basis. Then clearly

$$Y_B = B^{-1} \bar{P} \geq 0 \tag{2.11.4}$$

From the theory of linear program Y_B is optimal iff

$$e^T B^{-1} d(J) - 1 \leq 0 \tag{2.11.5}$$

$$\text{for all } J \subseteq N.$$

If Y_B is not optimal i.e., if Y_B does not satisfy (2.11.5), then a column $d(J)$ of D^T is chosen such that

$$e^T B^{-1} d(J) - 1 > 0 \tag{2.11.6}$$

Then the value of the objective function of DP(N) can be increased by introducing Y(J), satisfying (2.11.6), into the basis. In view of the fact that the coefficient matrix contains 2^n-1 columns, the problem of checking the criteria (2.11.5) and (2.11.6) leads to a formidable task. Column generation technique handles such problem by formulating as a suboptimization problem as follows;

$$\text{SP(N)}: \quad \begin{array}{l} \text{Maximize} \quad e^T B^{-1} X - 1 \\ \text{subject to } X \in \{d(J) \mid \phi \neq J \subseteq N\}. \end{array}$$

The suboptimization problem SP(N) is equivalent to a linear program where the aim is to maximize the linear function over the convex set C which is the convex hull of the set of points $\{d(J) \mid \phi \neq J \subseteq N\}$.

Theorem 2.8 shows that such a convex set is a convex polytope of diameter 1.

<u>Theorem 2.8</u>

Every d(J), for $J \neq \phi$ is an extreme point of C and every pair d(J'), d(J''), $J' \neq J''$ are adjacent extreme points in C.

<u>Proof</u> : The proof given here is based on the similar type of proof given in [12].

Let for any J, $|J| = k$, $1 \leq k \leq n$, the vector $f(J) \in R^{\frac{n(n+1)}{2}}$ be defined as

$$f(J) = (f_1, f_2, \ldots, f_n, f_{12}, \ldots, f_{1n}, f_{23}, \ldots, f_{n-1n})$$

such that $f_i = 1$ if $i \in J$,

$= 0$ otherwise,

$f_{ij} = -1$ if $i,j \in J$

$= M > 0$, otherwise with $M > 0$ arbitrary.

Clearly the inequality

$$f(J)^T X \leq \frac{k(k+1)}{2} \text{ holds for every } X \in \{d(S): \phi \neq S \subseteq N\}$$

and equality holds only if $X = d(J)$. Thus,

$$f(J)^T X = \frac{k(k+1)}{2}$$

is a supporting hyperplane to C which intersects it in d(J) and hence d(J) is an extreme point of C.

Next we shall show the adjacency among any pair of extreme points.

For a given pair of subsets J', J'', $J' \neq J''$ let us define the following,

$$S' = J' \cap J'', \quad S'' = J' \setminus J'', \quad S''' = J'' \setminus J'$$

with $|S'| = s_1$, $|S''| = s_2$, $|S'''| = s_3$, $|J'| = k_1$. We determine the vector $f(J',J'') \in R^{\frac{n(n+1)}{2}}$ as follows,

$$f(J',J'') = (f_1,f_2,\ldots,f_n,f_{12},\ldots,f_{1n},f_{23},\ldots,f_{n-1n})$$

where,

$f_i = 1$ if $i \in J'$,

$= \frac{s_2}{s_3}$ if $i \in S'''$,

$= 0$ otherwise,

and $\quad f_{ij} = -1 \quad \text{if} \quad i,j \in J'$

$$= -\frac{s_2}{s_3} \text{ if } j \in S''', \ i \in S'$$

$$= -\frac{s_2(s_2-1)}{s_3(s_3-1)} \text{ if } i,j \in S'''$$

$$= +M \text{ otherwise, with } M > k_1^2.$$

Let us consider the inequality

$$f(J',J'')^T X \leq \frac{k_1(k_1+1)}{2}$$

which holds for all $X \in \{d(S) : \phi \neq S \subseteq N\}$ and equality holds only if $X = d(J')$ or, $X = d(J'')$. Hence the hyperplane $f(J',J'') X = \frac{k_1(k_1+1)}{2}$ intersects C at two points namely $d(J')$, $d(J'')$. Thus $d(J')$, $d(J'')$ are adjacent in C.

This completes the proof.

Thus it is not fruitful to solve such a suboptimization problem by usual linear programming technique. This is so because the number of extreme points that are adjacent to any extreme point is exponentially large and any method trying to select the best among the adjacent points will require $o(2^n)$ number of computations.

Next we show that SP(N) can be considered as maximization of a quadratic Pseudo-Boolean function.

Let

$$e^T B^{-1} = (q_1, q_2, \ldots, q_n, q_{12}, \ldots, q_{n-1n}) \in R^{\frac{n(n+1)}{2}}.$$

Define $u \in R^n$, for any

$X \in \{d(J) \mid J \subseteq N\}$ as $u_i = x_i$, then clearly $u_i u_j = x_{ij}$. Then SP(N) is equivalent to

$$\text{SP1(N)}: \quad \text{Max} \sum_{i=1}^{n} q_i u_i + \sum_{1 \leq i < j \leq n} q_{ij} u_i u_j$$

$$u_i = 0 \text{ or, } 1, \quad 1 \leq i \leq n.$$

No special purpose technique to solve SP1(N) is discussed here. In order to solve the DP(N) by column generation technique, it is required to solve SP1(N) several times. For large values of n, such an approach turns out to be inefficient.

Chapter III

EXTREME POINTS OF F(N)

3.1 *Introduction*

Study of the geometrical structure of the constraint polyhedron of a linear program is a very valuable information for developing its solution techniques. In general, the geometrical study of the polyhedron concerns with the structure of its extreme points and of their adjacency. Such an investigation is more important, if the linear program has a large number of extreme points. Keeping this in mind, study of the structure of F(N) is initiated in this chapter. Though we fail to apply the results of this chapter in computing the lower bound, we stress on the theoretical potentiality of these results.

This chapter mainly deals with the geometry of the convex polyhedron F(N). In particular, we explore the structure of its extreme points. First, one type of extreme points are identified. It is shown that such extreme points are members of V^*. The necessary and sufficient condition that any point of V^* is an extreme point of F(N), is also derived. Another type of extreme points of F(N) is characterized and it is shown that such points can be obtained from certain nonextreme point members of V^*. We believe that these two classes of extreme points characterise all the extreme points of F(N), but we are unable to prove it. Attempts made in this direction are reported

and in the process the class V_1 of Kounias and Marin, given in (1.5.1), is generalised to a broader class.

The adjacency structure of some of the known extreme points is also discussed in this chapter. It is shown that the points in $\bigcup_{i=1}^{8} V_i$, but not in the above two classes of extreme points, are not extreme points of $F(N)$.

We next introduce some preliminary concepts for immediate use in the subsequent sections.

3.2 *Priliminary concepts*

It is seen from (2.1.2) that there is a one-one correspondence between the family of non-empty subsets of $N = \{1,2,\ldots,n\}$ and the set of constraints of $LP(N)$. Thus without any ambiguity, we shall identify any constraint by the associated subset of N. Let S be a subset of N having s elements. We assume without loss of generality, that $S = \{1,2,\ldots,s\}$. Let T, J, I denote any subset of S,N, $S\setminus N$ respectively. The subscript attached to T, J, I will denote the cardinality of the subset. For instance, T_k denotes a subset of T of S for which $|T| = k$. Such a subscript is used when the cardinality of the set is needed for discussion and dropped when it is irrelevent. Generally Greek letters will be used for family of sets.

Let $\delta_k(S)$ denote the family of all subsets of S having k elements. For a given $I \subseteq N\setminus S$, let

$$I^k(S) = \{I \cup T_k | T_k \in \delta_k(S)\}.$$

Example 3.1 :

Let $N = \{1,2,3,4,5,6\}$, $S = \{1,2,3\}$.

Let $I = \{4\}$, then $I^2(S)$ is given by

$$\{4\}^2(S) = \{\{1,2,4\}, \{1,3,4\}, \{2,3,4\}.$$

Similarly when $I = \{4,5\}$,

$$\{4,5\}^2(S) = \{\{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}.$$

Now the set of constraints (2.1.2) is arranged in a rectangular array R(S) consisting of (s+1) rows and of 2^{n-s} columns. Each column corresponds to a subset $I \subseteq N\backslash S$ and the rows of R(S) will be counted as 0,1,2,...,s. The entry in k^{th} row and I^{th} column of R(S) contains all the constraints associated with the block $I^k(S)$. Thus each entry of R(S) contains ${}^sC_k$ number of constraints of (2.1.2). It is to be noted that $\phi^o(S)$ is empty.

Example 3.2 :

Let N, S be defined as in Example 3.1. Then the rectangular array R(S) is given by

k\I	ϕ	4	5	6	45	46	56	456
0	ϕ	(4)	(5)	(6)	(45)	(46)	(56)	(456)
1	(1)(2)(3)	(14) (24) (34)	(15) (25) (35)	(16) (26) (36)	(145) (245) (345)	(146) (246) (346)	(156) (256) (356)	(1456) (2456) (3456)
2	(12)(13) (23)	(124) (134) (234)	(125) (135) (235)	(126) (136) (236)	(1245) (1345) (2345)	(1246) (1346) (2346)	(1256) (1356) (2356)	(12456 (23456 (13456
3	(123)	(1234)	(1235)	(1236)	(12345)	(12346)	(12356)	(123456

Fig. 3.1

As in section 1.3, let

$$d(J) = (d_1, d_2, \ldots, d_n, d_{12}, \ldots, d_{1n}, d_{23}, \ldots, d_{n-1n}) \in R^{\frac{n(n+1)}{2}},$$

be a vector defined for each nonempty subset J of N. Given a $J \subseteq N$, the first n components of the vector $d(J)$, will be denoted by the vector

$$\underline{d}(J) = (d_1, d_2, \ldots, d_n) \in R^n.$$

Definition 3.1 : Ranks of a family of subsets

For a collection $\alpha(N)$ of subsets of N, we define rank of order two (rank of order one) as the maximum number of independent vectors in the set

$$\{d(J) \mid J \in \alpha(N)\} \quad (\{\underline{d}(J) \mid J \in \alpha(N)\})$$

and will be denoted by $\underline{r_2(\alpha(N))}$ $(\underline{r_1(\alpha(N))})$.

Clearly $r_2(\alpha(N)) \leq \frac{n(n+1)}{2}$, and $r_1(\alpha(N)) = n$. It is to be noted that whenever the collection $\alpha(N \setminus S)$ is referred the constraints under consideration are that of $LP(N \setminus S)$ i.e., the linear programming problem with respect to the events A_i, $i \in N \setminus S$. Similarly in this case, the vectors $d(I)$, $\underline{d}(I)$ are of dimensions $\frac{(n-s)(n-s+1)}{2}$, $(n-s)$ respectively.

To see the relation between $R(S)$, $\alpha(N \setminus S)$ and the set of extreme points of $F(N)$ let us consider the following examples.

Example 3.3 :

$N = \{1,2,3\}$, $S = \{1,2\}$. Then $R(S)$ has 3 rows and 2 columns.

	ϕ	(3)
0	ϕ	(3)
1	(1),(2)	(13),(23)
2	(12)	(123)

For n = 3, there are four extreme points of F(N) and all such extreme points are obtained from the following pair of points by permuting their indices [31].

x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	
1	1	1	1	1	1	(3.1.2)
1	1	0	1	0	0	(3.1.3)

The following are set of constraints that are satisfied as equalities atthe point (3.1.2)

$$\phi^1(S),\ \phi^2(S),\ \{3\}^0(S),\ \{3\}^1(S) \tag{3.1.4}$$

Similarly the constraints which are equalities at (3.1.3) are

$$\phi^1(S),\ \phi^2(S),\ \{3\}^1(S),\ \{3\}^2(S) \tag{3.1.5}$$

For other extreme points such set of constraints can be written by permuting the indices of N in (3.1.4) and (3.1.5).

Example 3.4 :

Let N = {1,2,3,4} . For n = 4 all the extreme points can be found from the following 5 points by permuting their indices.

x_1	x_2	x_3	x_4	x_{12}	x_{13}	x_{14}	x_{23}	x_{24}	x_{34}	
1	1	0	0	1	0	0	0	0	0	(3.1.6)
1	1	1	0	1	1	0	1	0	0	(3.1.7)
1	1	1	1	1	1	1	1	1	1	(3.1.8)
1	1	1	-2	1	1	-1	1	-1	-1	(3.1.9)
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	(3.1.10)

For $S = \{1,2\}$ the associated $R(S)$ is written as follows,

ϕ	(3)	(4)	(3,4)
(1)(2)	(13) (23)	(14) (24)	(134) (234)
(12)	(123)	(124)	(1234)

In terms of the entries of $R(S)$ the constraints which are equalities at the point (3.1.6) - (3.1.9), are respectively

$\phi^1(S), \phi^2(S), \{3\}^1(S), \{3\}^2(S), \{4\}^1(S), \{4\}^2(S), \{3,4\}^1(S), \{3,4\}^2(S)$

$\phi^1(S), \phi^2(S), \{3\}^0(S), \{3\}^1(S), \{4\}^1(S), \{4\}^2(S), \{3,4\}^0(S), \{3,4\}^1(S)$

$\phi^1(S), \phi^2(S), \{3\}^0(S), \{3\}^1(S), \{3\}^0(S), \{4\}^1(S), \{3,4\}^0(S), \{4\}^0(S)$

$\phi^1(S), \phi^2(S), \{3\}^0(S), \{3\}^1(S), \{4\}^2(S), \{3,4\}^1(S), \{3,4\}^2(S)$.

In a similar manner if S is taken to be N then the point (3.1.10) can be obtained by making the constraints $\phi^2(N)$, $\phi^3(N)$ as equalities.

From the above examples, it is observed that given an extreme point of $F(N)$, $n \leq 4$, a suitable S can be chosen such that

the associated $R(S)$ has the following properties.

(i) The extreme point satisfies either all the constraints of a particular block $I^k(S)$ as equalities or, none of the constraints of the block is satisfied as equality at the given point.

(ii) For a given $I \subseteq N\setminus S$, i.e., for a given column of $R(S)$ the extreme point satisfies at most two blocks as equalities. In other words, there are at most two integers k_1, k_2 such that the blocks $I^{k_1}(S)$ and $I^{k_2}(S)$ are equalities at the given extreme point, where $0 \leq k_i \leq s$, $i = 1,2,$. Further,

(a) if the point satisfies exactly two blocks of column I, then the blocks are consecutive i.e.,

$$|k_1-k_2| = 1.$$

and (b) if the point satisfies only one block of I then the block is either the first block or, the last block of the column of $R(S)$ i.e., either $k_1 = 0$, or, $k_1 = s$.

(iii) There exists a collection $\alpha(N\setminus S)$ of subsets of $N\setminus S$ such that

$$r_2(\alpha(N\setminus S)) = \frac{(n-s)(n-s+1)}{2},$$

and for each $I \in \alpha(N\setminus S)$ the associated column in $R(S)$ has exactly two consecutive blocks as equalities for the point under consideration.

In the next section we show that for any arbitrary n, we can obtain a class of extreme points of $F(N)$, by suitably

choosing S and following the rules (i), (ii) and (iii) which, in fact, have been observed for the case $n \leq 4$.

3.3 *Ext(1): Extreme points of first kind*

From definition 1.1, it is clear that any extreme point of F(N) is a solution to a system of linear equations obtained by considering some $\frac{n(n+1)}{2}$ independent constraints of (2.1.2) as equalities. The method adopted here to generate extreme points is to select a set of constraints from (2.1.2) such that this set contains a set of $\frac{n(n+1)}{2}$ linearly independent constraints as its subset and that it follows the rules (i), (ii) and (iii) for some given S. This set of constraints when considered as equalities gives rise to a feasible solution (obviously, unique) and hence, this solution is an extreme point of F(N).

This section contains 5 subsections. In the first three, we gradually develop the results which are useful to prove the main result of this chapter. The main theorem is given in 3.34. In 3.35, we generalise this result and give an illustration.

<u>3.31</u> In view of the rules (i) and (ii), here we consider the case when two consecutive blocks of column $\phi(S)$ of R(S) are equalities and derive a system which is equivalent to it.

Let, for some integer h, $1 \leq h \leq s-2$, $\phi^h(S)$, $\phi^{h+1}(S)$ be equalities. The system of equations so obtained is,

$$\sum_{i \in T_h} x_i - \sum_{\substack{i < j \\ i,j \in T_h}} x_{ij} = 1, \tag{3.3.1}$$

for all $T_h \in \delta_h(S)$, and

$$\sum_{i \in T_{h+1}} x_i - \sum_{\substack{i < j \\ i,j \in T_{h+1}}} x_{ij} = 1 \qquad (3.3.2)$$

for all $T_{h+1} \in \delta_{h+1}(S)$.

It is easily seen that system (3.3.1) occurs for $1 \leq h \leq s$ and system (3.3.2) occurs for $0 \leq h \leq s-1$. The combined system (3.3.1) - (3.3.2) occurs for $0 \leq h \leq s$.

Theorem 3.1 :

The system of linear equations (3.3.1) - (3.3.2) has the following point as the unique solution, for all values of h, $1 \leq h \leq s-2$;

$$x_i = \frac{2}{h+1}, \; x_{ij} = \frac{2}{h(h+1)}, \; i < j, \; i,j \in S \qquad (3.3.3)$$

Proof : By direct substitution of values of x_i, x_{ij} from (3.3.3) in (3.3.1) - (3.3.2), it is clearly seen that (3.3.3) is a solution of (3.3.1) - (3.3.2). It remains to show that this solution is unique. That is to show that among the total ${}^sC_h + {}^sC_{h+1}$ number of equations of the system there are $\frac{s(s+1)}{2}$ linearly independent equations. In theorem 4.7 of Chapter IV, existence of such a linearly independent subsystem of (3.3.1) - (3.3.2) is shown. For direct reference, an equivalent statement of theorem 4.7 is given here, by substituting N by S and J by T; " There exists a collection $\alpha(S)$ of subsets of S such that

$\alpha(S) \subseteq \delta_h(S) \cup \delta_{h+1}(S)$ and

$$r_2(\alpha(S)) = \frac{s(s+1)}{2}, \text{ for } 1 \le h \le s-2" .$$

This completes the proof.

Corollary 3.1 :

For $h = 0$ and $h = s-1$, although (3.3.3) is a solution of (3.3.1) - (3.3.2), the solution is not unique.

Proof : Since $|\delta_0(S) \cup \delta_1(S)| = |\delta_1(S)| = {}^sC_1 = s$. Hence for $h = 0$ the system has s linear equations in $\frac{s(s+1)}{2}$ variables, so its solution cannot be unique. Similarly,

$$|\delta_{s-1}(S) \cup \delta_s(S)| = {}^sC_{s-1} + {}^sC_s = s + 1,$$

so the system cannot have unique solution.

From theorem 3.1, it is clear that the system (3.3.1) - (3.3.2) is equivalent to the system (3.3.3).

3.32 Here we derive an equivalent system for the system of equations obtained when two consecutive blocks of any arbitrary column of R(S), considered as equalities, together with $\phi^h(S)$, $\phi^{h+1}(S)$.

Let us consider those points $X \in R^{\frac{n(n+1)}{2}}$ which besides being solution of (3.3.3) satisfy exactly two consecutive blocks of a particular column I of R(S), as equalities. Let the blocks which are equalities at the point under consideration be $I^\theta(S)$ and $I^{\theta+1}(S)$ for some $0 \le \theta \le s-1$. Substituting the values of x_i, x_{ij}, $i < j$, $i,j \in S$ from (3.3.3) we have,

$$\frac{2\theta}{h+1} + \sum_{i \in I} x_i - \sum_{\substack{i \in T_\theta \\ j \in I}} x_{ij} - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} - \frac{\theta(\theta-1)}{h(h+1)} = 1, \qquad (3.3.4)$$

for all $T_\theta \in \delta_\theta(S)$.

Similarly,

$$\frac{2(\theta+1)}{h+1} + \sum_{i \in I} x_i - \sum_{\substack{i \in T_{\theta+1} \\ j \in I}} x_{ij} - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} - \frac{\theta(\theta+1)}{h(h+1)} = 1 \tag{3.3.5}$$

for all $T_{\theta+1} \in \delta_{\theta+1}(S)$.

Lemma 3.1 :

The system of equations (3.3.4) and (3.3.5) is equivalent to the system of equations (3.3.4) and (3.3.6), where (3.3.6) is

$$\sum_{j \in I} x_{ij} = \frac{2(h-\theta)}{h(h+1)}, \text{ for all } i \in S. \tag{3.3.6}$$

Proof : For any $T_{\theta+1}$ select $T_\theta \subseteq T_{\theta+1}$ and let $k \in T_{\theta+1} \setminus T_\theta$. Then (3.3.5) for each $T_{\theta+1}$ can be written as,

$$\frac{2(\theta+1)}{(h+1)} + \sum_{i \in I} x_i - \frac{\theta(\theta+1)}{h(h+1)} - \sum_{\substack{i \in T_{\theta+1} \\ j \in I}} x_{ij} - \sum_{j \in I} x_{kj}$$

$$- \sum_{\substack{i < j \\ i,j \in I}} x_{ij} = 1 \tag{3.3.7}$$

Subtracting (3.3.7) from (3.3.4) we get,

$$\sum_{j \in I} x_{kj} = \frac{2(h-\theta)}{h(h+1)}.$$

Such an equivalence relation can be obtained for each of the constraints in (3.3.5) and hence (3.3.4) - (3.3.5) is equivalent to the system (3.3.4) together with

$$\sum_{j \in I} x_{ij} = \frac{2(h-\theta)}{h(h+1)}, \quad \forall \; i \in S$$

This completes the proof.

3.33 Here we derive conditions such that any block R(S) can be represented by a single constraint. Once the blocks are reduced to single constraints, it is also shown that every column of R(S) has a dominant constraint in the following sense.

Definition 3.2 : Dominant constraint.

A constraint in a set of constraints is said to be a dominant constraint of the set, if

(i) this constraint is satisfied then the rest of the constraints are automatically satisfied and

(ii) a point satisfies any constraint of the set as equality, then the dominant constraint is also equality at the point and otherwise, the point is not feasible to the set of constraints.

We now prove lemma 3.2 which gives the condition that any block is reduced to a single constraint.

Lemma 3.2 :

Let $\alpha(N\backslash S)$ be a collection of subsets of $N\backslash S$ such that

$$r_1(\alpha(N\backslash S)) = n-s \tag{3.3.8}$$

and let there be integers h, a_i, $i \in N\backslash S$ such that

$$0 \leq h - \sum_{i \in I} a_i \leq s-1, \text{ for all } I \in \alpha(N\backslash S).$$

Then each of the blocks of R(S) is reduced to a single constraint if the constraints

$$\phi^h(S),\ \phi^{h+1}(S),\ I^{\theta(I)}(S),\ I^{\theta(I)+1}(S) \tag{3.3.9}$$

are made equalities for all $I \in \alpha(N\setminus S)$ where

$$\theta(I) = h - \sum_{i \in I} a_i \tag{3.3.10}$$

Proof : By making the constraints $I^{\theta(I)}(S)$, $I^{\theta(I)+1}(S)$ equalities for $I \in \alpha(N\setminus S)$ together with $\phi^h(S)$, $\phi^{h+1}(S)$, the system of equations so obtained is equivalent to,

$$(3.3.3) - (3.3.4), \text{ for all } I \in \alpha(N\setminus S)$$

$$(3.3.6), \text{ for all } I \in \alpha(N\setminus S).$$

From (3.3.6), we have

$$\sum_{i \in I} x_i = \frac{2(h-\theta(I))}{h(h+1)}, \quad \forall\, I \in \alpha(N\setminus S) \text{ and } \forall\, i \in S.$$

Since $r_1(\alpha(N\setminus S)) = n-s$, the integers a_i, $i \in N\setminus S$ are uniquely determined by (3.3.10). Hence we have,

$$\Longrightarrow x_{ij} = \frac{2a_i}{h(h+1)}, \quad \forall\, j \in N\setminus S, \text{ and } \forall\, i \in S. \tag{3.3.11}$$

Now let us consider any block $I^k(S)$ for any $I \subseteq N\setminus S$. By substituting the values x_i, x_{ij} from (3.3.3) and (3.3.11), it is easily seen that the whole set of inequalities in $I^k(S)$ reduces to a single inequality given by,

$$\frac{2k}{h+1} + \sum_{i \in I} x_i - \frac{k(k-1)}{h(h+1)} - k \sum_{i \in I} \frac{2a_i}{h(h+1)} - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} \leq 1. \tag{3.3.12}$$

This completes the proof.

In theorem 3.2 we show the existence of the dominant constraints.

<u>Theorem 3.2</u> :

The set of constraints associated with any column I of R(S) has a single dominant constraint under the condition (3.3.3) and (3.3.11).

<u>Proof</u> : From lemma 3.1, the condition (3.3.3) and (3.3.11) reduce any block to a single constraint and this constraint with reference to block $I^m(S)$; is given by, from (3.3.12);

$$\sum_{i \in I} x_i - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} \leq 1 - \frac{2m}{h+1} + \frac{m(m-1)}{h(h+1)} + m \sum_{i \in I} \frac{2a_i}{h(h+1)} \tag{3.3.13}$$

Let RH(m) denote the right hand side of (3.3.13) i.e.

$$RH(m) = 1 - \frac{2m}{h+1} + \frac{m(m-1)}{h(h+1)} + m \sum_{i \in I} \frac{2a_i}{h(h+1)} .$$

Then we have,

$$RH(m) - RH(m+1)$$

$$= \frac{2}{h+1} - \frac{2m}{h(h+1)} - \sum_{i \in I} \frac{2a_i}{h(h+1)}$$

$$= \frac{2}{h(h+1)} (h - m - \sum_{i \in I} a_i).$$

Thus,

$$\left.\begin{array}{ll} \text{(i)} & \text{for } m < h - \sum_{i \in I} a_i, \quad RH(m) > RH(m+1) \\ \text{(ii)} & \text{for } m > h - \sum_{i \in I} a_i, \quad RH(m) < RH(m+1) \\ \text{(iii)} & \text{for } m = h - \sum_{i \in I} a_i, \quad RH(m) = RH(m+1) . \end{array}\right\} \tag{3.3.14}$$

Thus RH(m) takes its minimum value at

$$m = \theta(I) = h - \sum_{i \in I} a_i, \text{ and at } m = \theta(I) + 1.$$

For a fixed I, left hand side of (3.3.11) remains constant for different values of m and hence we have following two cases.

Case (i) : If $0 \leq h - \sum_{i \in I} a_i \leq s-1$, then clearly the inequality (3.3.13) corresponding to $m = \theta(I)$ or, $\theta(I) + 1$ is dominating among $I^m(S)$, $1 \leq m \leq s$.

Case (ii) :

(a) If $h - \sum_{i \in I} a_i > s-1$, then clearly the last block of the column I(S) is dominating over all the constraints of the column. That is $I^s(S)$ is the dominant constraint.

(b) If $h - \sum_{i \in I} a_i < 1$, using same argument $I^0(S)$ is the dominant column of the set of constraints I(S).

This completes the proof.

Remark : In theorem 3.2, it is shown that under the condition (3.3.3) and (3.3.11), the dominant constraint of a column appears in exactly one block for some column and for others it appears in two consecutive blocks. This result helps to consider the rule (ii) mentioned in section 3.2, for generation of extreme points.

3.34 In this section we give the main result of the chapter. By using the results obtained in sections 3.31 - 3.33, we derive conditions to generate aclass of extreme points.

Since all the results of sections 3.31 – 3.33 are obtained for a specified S, there is no loss of generality by dropping the suffix S from $I^{\theta(I)}(S)$. We henceforth, use $I^{\theta(I)}$ instead of $I^{\theta(I)}(S)$.

Theorem 3.3 :

Let the integers $h \geq 1$ and a_i, $i \in N \backslash S$ be chosen such that they satisfy the following conditions;

(i) $1 \leq h \leq s-2$

(ii) there exists a collection of subsets, $\alpha(N \backslash S)$ such that

$$r_2(\alpha(N \backslash S)) = |\alpha(N \backslash S)| = \frac{(n-s)(n-s+1)}{2}$$

(iii) $0 \leq \theta(I) \leq s-1$,

$$\forall\, I \in \alpha(N \backslash S) \text{ where } \theta(I) = h - \sum_{i \in I} a_i.$$

Then the solution of the constraints

$$\phi^h,\ \phi^{h+1},\ I^{\theta(I)},\ I^{\theta(I)+1},\ \forall\, I \in \alpha(N \backslash S) \tag{3.3.15}$$

considered as equalities, is an extreme point of F(N).

Proof : If $r_2(\alpha(N \backslash S)) = \frac{(n-s)(n-s+1)}{2}$,

then $r_1(\alpha(N \backslash S)) = n-s$.

Hence, there exists a subcollection

$$\beta(N \backslash S) \subseteq \alpha(N \backslash S),\ |\beta(N \backslash S)| = n-s$$

such that $r_1(\beta(N \backslash S)) = n-s$.

From the earlier results, when the constraints ϕ^h, ϕ^{h+1}, $I^{\theta(I)}$, $I^{\theta(I)+1}$, $\forall\, I \in \beta(N\setminus S)$ are made equalities, the whole set of constraints is equivalent to (3.3.3), (3.3.11) and the following set of inequalities.

$$\sum_{i \in I} x_i - \sum_{\substack{i<j\\ i,j \in I}} x_{ij} \leq \begin{cases} RH(\theta(I)) = RH(\theta(I)+1) \\ \qquad\qquad \text{for all } I \text{ with } 0 \leq \theta(I) \leq s-1 \\ RH(0), \text{ for all } I \text{ with } \theta(I) < 0 \\ RH(s), \text{ for all } I \text{ with } \theta(I) > s-1. \end{cases}$$

Hence the system of equations, obtained by making all the constraints of (3.3.15), equalities, is equivalent to

(3.3.3), (3.3.11) and

$$\sum_{i \in I} x_i - \sum_{\substack{i<j\\ i,j \in I}} = RH(\theta(I)) = RH(\theta(I)+1) \text{ for all } I \in \alpha(N\setminus S) \tag{3.3.16}$$

Since $r_2(\alpha(N\setminus S)) = \dfrac{(n-s)(n-s+1)}{2}$,

(3.3.16) has at most one solution. We show that (3.3.16) has exactly one solution which is

$$x_i = \frac{(2h+1)\, a_i - a_i^2}{h(h+1)}, \quad x_{ij} = \frac{2a_i\, a_j}{h(h+1)},$$

$$\text{for all } i < j,\ i,j \in (N\setminus S). \tag{3.3.17}$$

It is not difficult to check that (3.3.17) satisfies (3.3.16). Hence the system of equations obtained from (3.3.15) has a unique solution, given by

(3.3.3), (3.3.11) and (3.3.17).

The unique point so obtained is an extreme point if and only if it is a feasible solution of $F(N)$. It is merely a routine work to check that the point X given by (3.3.3), (3.3.11), (3.3.17) is the point (a,h) in V^* where

$$a_i = 1,\ i \in S \text{ and } a_i,\ i \notin S \text{ are predetermined}$$

and hence from theorem 2.1, X is feasible.

This completes the proof.

Remark 1 : It is easy to see that the restriction on $\theta(I)$ in the assumption (iii) of theorem 3.3 is to ascertain that both the blocks $I^{\theta(I)}$, $I^{\theta(I)+1}$ are existing in R(S), for all $I \in \alpha(N\setminus S)$. Since relation (3.3.16) requires only the existence of the dominant constraint, it is sufficient to have atleast one of these two blocks where the dominant constraint appears. Thus it is sufficient to have the restriction

$$-1 \leq \theta(I) \leq s.$$

But from lemma 3.2 in order that the set of constraints are reduced to a set of dominant constraints it is necessary that for some subsets I, the consecutive blocks $I^{\theta(I)}$, $I^{\theta(I)+1}$ are equalities. Hence the hypothesis (iii) of theorem 3.3 can be relaxed a little as follows;

(iii)' $\quad -1 \leq \theta(I) \leq s \quad \forall I \in \alpha(N\setminus S)$

and there is a subcollection $\beta(N\setminus S) \subseteq \alpha(N\setminus S)$ such that $|\beta(N\setminus S)| = r_1(\beta(N\setminus S)) = n-s$ and $0 \leq \theta(I) \leq s-1,\ \forall I \in \beta(N\setminus S)$.

Remark 2 : Methods to generate the extreme point by using theorem 3.3 can be summarized as follows; first a subset S and integers $h \geq 1$, a_i, $i \in N\backslash S$ are chosen. If these integers satisfy the hypotheses of theorem 3.3, then the point

$X = (1,1,\ldots,1, a_{s+1}, a_{s+2},\ldots,a_n, h) \in V^*$ is an extreme point of P(N). Thus for all possible choices of S such that $|s| \geq 2$ and for all possible choices of integer $h \geq 1$, a_i satisfying the conditions (i), (ii), (iii) of theorem 3.3, a class of extreme points can be generated and it can be shown that such extreme points are members of V^*. The class of such extreme points will be denoted by Ext (1). From theorem 3.3, following results can be directly derived.

Corollary 3.1 :

For $n = 1,2,3,4$ all the extreme points of P(N) are of first kind.

Proof : From the examples discussed in section 3.2 the proof can be directly derived.

Corollary 3.2 :

$$V_2 \cup V_3 \subseteq \text{Ext } (1).$$

Proof : Let $S = J_r$ and $a_i = 0$ for all $i \in N\backslash S$. Then define $\alpha(N\backslash S) = \{\{i\}, \{i,j\} \mid i < j,\ i,j \in N\backslash S\}$. Clearly for $1 \leq h \leq r-2$, the hypotheses of theorem 3.3 are satisfied and from (1.5.2) such a point is a member of V_2.

Similarly by taking $S = J_{n-1}$, $J_1 = N\backslash S$, $a_i = -1$, $i \in J_1$ and defining $\alpha(N\backslash S) = \{\{i\}, i \in J_1\}$, it is not hard to check that the hypotheses of theorem 3.3 are satisfied and from (1.5.3) such a point is a member of V_3.

3.35 Thus so far a subset of V^* is classified such that the points are extreme points of the convex polyhedron $F(N)$. The question that still remains unanswered is that whether there is any other extreme point in V^*. The following theorem gives the necessary and sufficient condition for a point of V^* to be an extreme point of $F(N)$.

Theorem 3.4 :

A point $(a,h) \in V^*$ is an extreme point of the convex polyhedron $F(N)$ if and only if there exists a collection $\alpha(N)$ such that,

(i) $r_2(\alpha(N)) = \dfrac{n(n+1)}{2}$,

(ii) $\theta(J) = 0$ or, -1 for all $J \in \alpha(N)$.

Proof : Substituting the values of x_i and x_{ij} interms of a_i, h in any of the constraints (2.1.2) we have, from

$$\left\{ h - \sum_{i \in J} a_i \right\} \left\{ h+1 - \sum_{i \in J} a_i \right\} \geq 0.$$

For this constraint to be equality at the point $X = (a,h) \in V^*$ it is necessary and sufficient that

$$\left\{ h - \sum_{i \in J} a_i \right\} \text{ is either 0 or, } -1$$

$$\Longrightarrow \quad \theta(J) = 0 \text{ or, } -1. \qquad (3.3.18)$$

Thus $X \in (a,h) \in V^*$ is an extreme point iff there are $\frac{n(n+1)}{2}$ linearly independent constraints which are equalities at X. Since $\alpha(N)$ provides requisite number of linearly independent constraints and condition on $\theta(J)$, $J \in \alpha(N)$ ensures that these independent constraints are equalities, the rest of the proof immediately follows.

The difference between theorem 3.3 and theorem 3.4 is apparent. The later is the generalization of the former in the sense that, in the former a subset S of indices is distinguished for which the integers a_i are fixed at the level 1 and on the other hand there is no such restriction in theorem 3.4. Another remarkable difference between these two theorems is that there are examples to establish the result of theorem 3.3 whereas no extreme point can be found which satisfies the condition of theorem 3.4 but not of theorem 3.3.

Example 3.5 :

Let $N = \{1,2,3,4,5,6,7,8\}$

$S = \{1,2,3,4\}$

Then the associated R(S) can be written as,

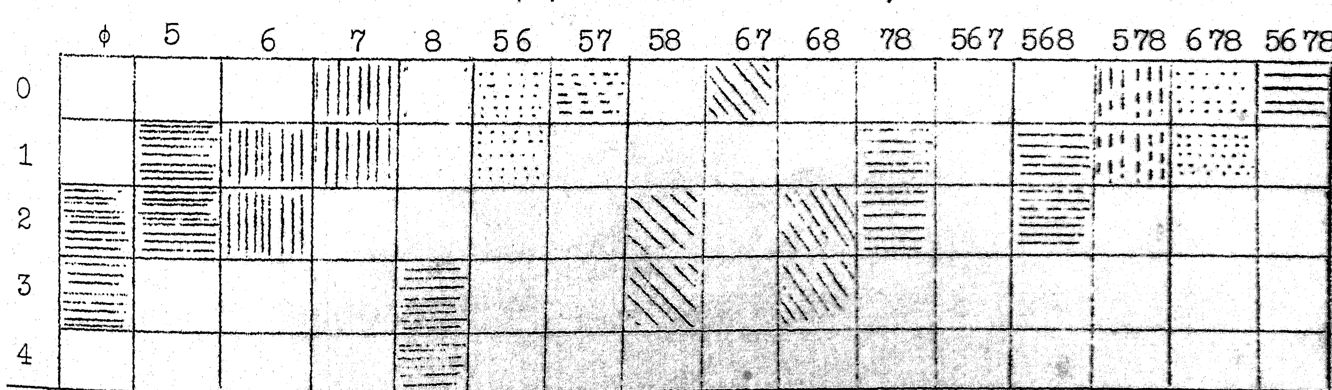


Fig. 3.4

Let $a_5 = 1$, $a_6 = 1$, $a_7 = 2$, $a_8 = -1$ and $h = 2$. For any $I \subseteq N\backslash S$, the blocks corresponding to $I^{\theta(I)}$ and $I^{\theta(I)+1}$ are identified as the shaded blocks in Fig. 3.4.

The condition (i) of theorem 3.3 is directly satisfied as $0 \leq h \leq s-2$. From the figure the collection $\alpha(N\backslash S)$ can be derived as

$$\alpha(N\backslash S) = \{\{i\},\ \{i,j\},\ \text{for all } i < j,\ i,j \in N\backslash S\}.$$

i.e. $\alpha(N\backslash S) = \{\{5\},\{6\},\{7\},\{8\},\{5,6\},\{5,7\},\{5,8\},\{6,7\},\{6,8\},\{7,8\}\}$

$$r_2(\alpha(N\backslash S)) = 10, \quad r_1(\alpha(N\backslash S)) = 4.$$

We shall show that the condition (iii)' of theorem 3.3 is satisfies for such $\alpha(N\backslash S)$.

It is not difficult to check that

$$-1 \leq \theta(I) \leq s \quad \text{for all } I \in \alpha(N\backslash S).$$

It is to be noted here that had the above condition been violated, there would not have any shaded region for the corresponding subsets I.

We define

$$\beta(N\backslash S) = \{\{5\},\{6\},\{7\},\{8\}\}, \text{ then } r_1(\beta(N\backslash S)) = 4.$$

It is also seen that

$$0 \leq \theta(I) \leq s-1, \text{ for all } I \in \beta(N\backslash S)$$

Such condition ensures that for every $I \in \beta(N\backslash S)$ there is a pair of consecutive shaded blocks in the figure 3.4.

Thus the point

$$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2,\ -1,\ h = 2) \in V^* \text{ is an}$$

extreme point of F(N).

It is also easy to check that this extreme point is not a member of any of the 8 classes of vertex points generated by Kounias and Marin [31].

<u>Remark</u> : It is observed that in theorem 3.3, the number of constraints which are made equalities in fact are more than the required number $\frac{n(n+1)}{2}$. It is not difficult to find exactly $\frac{n(n+1)}{2}$ constraints from (3.3.15), which when made equalities, give rise to the extreme point. These constraints can be picked up for the above example as follows;

$\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\}$
$\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\},$
$\{1,2,5\},\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{1,2,6\},\ \{7\},\{1,7\},$
$\{2,7\},\{3,7\},\{4,7\},\{1,2,3,8\},\{1,2,4,8\},\{1,3,4,8\},\{2,3,4,8\},$
$\{1,2,3,4,8\},\{5,6\},\{5,7\},\{1,2,5,8\},\{6,7\},\{1,2,6,8\},\{1,7,8\}.$

It is merely a routine work to see that the coefficient matrix, with respect to the above set of constraints, is non-singular and that it is a subset of the set of constraints (3.3.15).

It is to be noted that such choice of $\frac{n(n+1)}{2}$ independent constraints from (3.3.15) is not unique and the choice of $\alpha(N\setminus S)$ is also not unique. Thus there are several ways to pick up $\frac{n(n+1)}{2}$ linearly independent constraints to define the same extreme

point. Such situation refers to degeneracy. The high rate of degeneracy of the extreme points makes it difficult to solve LP(N) by simplex method.

3.4 Ext(2) : extreme points of second kind

All the extreme points of F(N) are not covered by the class Ext(1). It is observed that there exists another class of extreme points which are not members of V*. The main aim of this section is to show the existence of a class of extreme points of F(N), which are not members of V* but can be generated from certain nonextreme point members of V*. Assume that $X = (a,h) \in V^*$ is not an extreme point of F(N). Let $\alpha(N)$ be the class of subsets of N corresponding to which the associated constraints are equalities at X. Clearly from theorem 3.4, we have,

$$r_2(\alpha(N)) < \frac{n(n+1)}{2} .$$

The purpose of this section is to choose some more constraints which are not in $\alpha(N)$ but which together with $\alpha(N)$ constitute the class $\bar{\alpha}(N)$ such that

$$r_2(\bar{\alpha}(N)) = \frac{n(n+1)}{2}$$

and when all the constraints in $\bar{\alpha}(N)$ are made equalities they give rise to an extreme point of F(N).

For a given $S \subseteq N$ and for given integers h, a_i, $i \in N\backslash S$ where $1 \leq h \leq s-1$, let the subsets of $N\backslash S$ be partitioned as;

$$\pi = \{I \subseteq N\setminus S \mid -1 \leq \theta(I) \leq s\},$$

$$\pi_1 = \{I \subseteq N\setminus S \mid \theta(I) < -1\},$$

$$\pi_2 = \{I \subseteq N\setminus S \mid \theta(I) > s\},$$

where $\theta(I) = h - \sum_{i \in I} a_i$.

From theorem 3.3, it is clear that if there exists

$\alpha(N\setminus S) \subseteq \pi$ such that

$$|\alpha(N\setminus S)| = \frac{(n-s)(n-s+1)}{2}$$

$$r_2(\alpha(N\setminus S)) = \frac{(n-s)(n-s+1)}{2}$$

and there exists a $\beta(N\setminus S) \subseteq \alpha(N\setminus S)$, $|\beta(N\setminus S)| = r_1(\beta(N\setminus S)) = (n-s)$ such that $\theta \leq \theta(I) \leq s-1 \ \forall\, I \in \beta(N\setminus S)$ then the point $(1, 1, \ldots 1, a_{s+1} \ldots a_n, h) \in V^*$ is an extreme point of $F(N)$.

Here we shall consider the case when $r_2(\pi) < \frac{(n-s)(n-s+1)}{2}$ but there exists a

$\beta(N\setminus S) \subseteq \pi$ such that $|\beta(N\setminus S)| = r_1(\beta(N\setminus S)) = n-s$.

Following arguments of section 3.31 - 3.34 it can be shown that the whole set of constraints are equivalent to

$$\left.\begin{array}{l} x_i = \dfrac{2}{h+1}\,,\ x_{ij} = \dfrac{2}{h(h+1)} \quad i < j,\ i,j \in S \\ x_{ij} = \dfrac{2a_i}{h(h+1)}\ i \in N\setminus S,\ j \in S \end{array}\right\} \qquad (3.4.1)$$

$$\sum_{i \in I} x_i - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} \leq \begin{cases} RH(0) \quad \forall\, I \in \pi_1 & (3.4.2) \\ RH(\theta(I)) = RH(\theta(I)+1), \forall I \in \pi & (3.4.3) \\ RH(s) \quad \forall\, I \in \pi_2 & (3.4.4) \end{cases}$$

By direct substitution of values of x_i, x_{ij} in terms of a_i, h, it can be shown that the point

$$(1, 1, \ldots, 1, a_{s+1}, \ldots, a_n, h) \in V^*$$

satisfies the system (3.4.1) - (3.4.4) with (3.4.3) being equalities that is,

$$\sum_{i \in I} x_i - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} = RH(\theta(I)) = RH(\theta(I)+1), \qquad (3.4.3)'$$

for all $I \in \pi$.

The system (3.4.1), (3.4.2), (3.4.3)' and (3.4.4) defines a convex polyhedron which is a subset of F(N) and hence any point of this convex polyhedron is also a feasible point of F(N). Based on this observation we prove theorem 3.5.

<u>Theorem 3.5</u> :

Any extreme point of the convex polyhedron defined by the system (3.4.1), (3.4.2), (3.4.3)' and (3.4.4) is also an extreme point of F(N).

<u>Proof</u> : Obviously any extreme point of such polyhedron satisfies the equations (3.4.1) and (3.4.3)' and there are at least 't' number of independent constraints among (3.4.2) and (3.4.4) which are equalities at the extreme point, where

$$t = \frac{(n-s)(n-s+1)}{2} - r_2(\pi).$$

These t constraints together with the constraints of π form a collection $\bar{\pi}$ such that $r_2(\bar{\pi}) = \frac{(n-s)(n-s+1)}{2}$. In order to

show that this point is an extreme point of $F(N)$ it is necessary to show that the point satisfies $\frac{n(n+1)}{2}$ linearly independent constraints of (2.1.2) as equalities. Following the method adopted to prove theorem 3.2, it can be shown that the set of constraints

$$\phi^h, \ \phi^{h+1}, \ I^{\theta(I)}, \ I^{\theta(I)+1}, \ \forall I \in \pi$$

and $$I^o, \ \forall I \in \bar{\pi} \cap \pi_1$$

$$I^s, \ \forall I \in \bar{\pi} \cap \pi_2$$

yield the required number of independent constraints.

This completes the proof.

Theorem 3.6 :

There exists at least one extreme point of (3.4.1), (3.4.2), (3.4.3)' and (3.4.4).

Proof : Consider the convex polyhedron F_1 defined by the following system;

the system (3.4.1) and (3.4.3)' together with

$$\sum_{i \in I} x_i - \sum_{\substack{i < j \\ i,j \in I}} x_{ij} \leq RH(\theta(I)) \qquad (3.4.5)$$

for all $I \in \pi_1 \cup \pi_2$.

The point

$$X = (1,1,1,\ldots 1, \ a_{s+1}, \ a_s, \ldots, a_n, h) \in V^* \text{ satisfies}$$

all the constraints of (3.4.5) as equalities and hence is an extreme point of F_1.

It is clear that the system of inequalities (3.4.5) defers from the system (3.4.2), (3.4.4) by the right hand side constants.

It can be easily derived from a corollary of well known Resolution Theorem [39, page 119], [18], [48], [50], [51] that any convex polyhedron which is defined by the system (3.4.5) by changing right hand side vector, has at least one extreme point as long as the polyhedron has at least one feasible point.

Since the system (3.4.1), (3.4.2), (3.4.3)' and (3.4.4) has a feasible point, namely X, it has at least one extreme point.

This completes the proof.

As the theorem 3.4 gives the necessary as well as sufficient condition for a point in V* to be an extreme point of F(N). Clearly any extreme point obtained by theorem 3.6, which is shown to be an extreme point of F(N) in theorem 3.5, cannot be a member of V*. Such extreme points are said to be extreme point of 2nd kind. Let the class of extreme points of 2nd kind be denoted by Ext (2).

Example 3.6 :

As in Ex 3.5, let

$$N = \{1,2,3,4,5,6,7,8\},\ S = \{1,2,3,4\},$$

let $h = 2$, $a_5 = 1$, $a_6 = 1$, $a_7 = 2$ and $a_8 = 2$

$$\pi = \{(5), (6), (7),(8),(56),(57),(58),(67),(68)\}.$$

Clearly $\quad r_2(\pi) = \frac{(n-s)(n-s+1)}{2} - 1.$

The reduced set of constraints is given by

$$x_{78} \geq 1,$$

$$x_{78} \geq \frac{1}{3},$$

$$x_{78} \geq -\frac{2}{3}.$$

Clearly above system has an extreme point as $x_{78} = 1$. Hence the extreme point of $F(N)$ is

$$x_i = \frac{2}{3}, \ 1 \leq i \leq 6, \ x_7 = 1, \ x_8 = 1$$

$$x_{ij} = \frac{1}{3}, \ 1 \leq i < j \leq 6, \ x_{i7} = x_{i8} = \frac{2}{3}, \ x_{78} = 1.$$

Corresponding $X \in V^*$ differs from above by $x_{78} = 4/3$.

Thus we see that for an $(a,h) \in V^*$ with $a_i = 1$, $i \in S$ for some subset S if there exists a collection $\beta(N \setminus S)$ such that

$$r_1(\beta(N \setminus S)) = n-s \quad \text{and}$$

$$0 \leq \theta(I) \leq s-1 \quad \text{for all } I \in \beta(N \setminus S)$$

then either the point $(a,h) \in V^*$ is an extreme point or, an extreme point can be obtained from $(a,h) \in V^*$ which satisfies $I^{\theta(I)}$, $I^{\theta(I)+1}$, as equalities, $\forall\, I \in \beta(N \setminus S)$. The following theorem gives the generalisation of this result.

Theorem 3.7 :

For any given extreme point X of $F(N)$ there exists a point $(a,h) \in V^*$ such that (a,h) and X satisfy at least n independent constraints as equalities in common. Conversely,

given any $(a,h) \in V^*$ which satisfies at least n constraints, which have rank of order one as n, as equalities then there exists an extreme point X of F(N) such that (a,h) and X have at least n common equality constraints.

Proof : For any extreme point $X \in F(N)$, let $\alpha(N)$ be the set of constraints that are equalities at X. Hence

$$r_2(\alpha(N)) = \frac{n(n+1)}{2} ,$$

$$r_1(\alpha(N)) = n.$$

Let $\beta(N) \subseteq \alpha(N)$, $|\beta(N)| = n$ and $r_1(\beta(N)) = n$. We claim that there always exists a set of (n+1) integers $a_1, a_2, \ldots, a_n, h$ such that they satisfy the following

$$\sum_{i \in I} a_i = h \text{ or, } h+1, \quad \forall\, I \in \beta(N) \qquad (3.4.6)$$

Because the system (3.4.6)is a system of n linear equations with n+1 variables, by suitable choice of h, n integers a_i can be determined. Hence the point $(a,h) \in V^*$ defined by these integers satisfies all the constraints J, $J \in \beta(N)$ for $\theta(J) =$

$$h - \sum_{i \in J} a_i = 0, \ -1.$$

Conversely, let $\alpha(N)$ be the set of constraints which are equalities at $(a,h) \in V^*$. If $r_2(\alpha(N)) = \frac{n(n+1)}{2}$, then the theorem boils down to result obtained in theorem 3.4. Let $r_2(\alpha(N)) < \frac{n(n+1)}{2}$, then by making use of theorem 3.6, some more constraints can be chosen which together with $\alpha(N)$ constitute $\bar{\alpha}(N)$ such that $r_2(\bar{\alpha}(N)) = \frac{n(n+1)}{2}$, and when the constraints of $\bar{\alpha}(N)$ are made

equalities they give rise to an extreme point of F(N).

This completes the proof.

3.5 *Characterisation of extreme points of F(N)*

In this section we will describe some attempts made to find all the extreme points of the convex polyhedron F(N). It is believed that Ext (1) together with Ext (2) exhaust all the extreme points of F(N). The following discussion is aimed to strengthen the belief that all the extreme points of F(N) can be generated from the extreme points of F(S), $S \subseteq N$ except possibly with one exception.

Let N-t, $t > 0$ denote any subset of N having n-t elements. Without any loss of generality we shall assume that N-t = $\{1,2,\ldots,n-t\}$. Clearly F(N-t) is contained in $R^{\frac{(n-t)(n-t+1)}{2}}$. Any point $X^{n-t} \in R^{\frac{(n-t)(n-t+1)}{2}}$ can be defined similarly as in the case of $X \in R^{\frac{n(n+1)}{2}}$.

For the sake of uniformity we shall denote $X \in R^{\frac{n(n+1)}{2}}$ by X^n. Clearly X^n and X^{n-t} have some coordinates in common, namely

$$x_i^n = x_i^{n-t}, \; x_{ij}^n = x_{ij}^{n-t}, \text{ for } i < j, \; i,j \in N\text{-}t.$$

In this section it is shown that given an extreme point X^{n-t} of F(N-t) several extreme points of F(N) can be obtained such that these extreme points have same values for the coordinates

theory for generating extreme points of $F(N)$, is of interest of its own. In [31] the class of points V_1 given in (1.5.1), is obtained from the vertex points of $F(N-t)$ by assigning the value zero to the new variables, arising due to the increase in the dimension. Here it is shown that such extension from $F(N-t)$ to $F(N)$ can be generalised by putting several values, besides zero, to the new variables. We consider two different cases, namely when X^{n-t} is an extreme point of first kind and when X^{n-t} is an extreme point of second kind. In 3.53, we show that there is one extreme point of $F(N)$ which cannot be obtained from any extreme point of $F(N-t)$. We believe that this is the only exception possible. We give a conjecture in this regard. Such a belief is based on both theoretical and computational experiences. All the extreme points generated by random computation are members of either Ext (1) or, Ext (2).

3.51 : In this section we consider the possible extension of X^{n-1} to extreme points of $F(N)$ where X^{n-1} is an extreme point of first kind of $F(N-1)$.

Since X^{n-1} is an extreme point of first kind, there exists a subset $S \subseteq N-1$ and integers $h \geq 1$, a_i, $i \in N-1\setminus S$ such that X^{n-1} satisfies the following set of constraints as equalities,

$$\phi^{h},\ \phi^{h+1},\ I^{\theta(I)},\ I^{\theta(I)+1}\ ,\ \text{for all } I \in \alpha(N-1\setminus S).$$

Since $S \subseteq N-1 \implies S \subseteq N$, the set of constraints of $LP(N)$ can be arranged as $R(S)$. The columns of $R(S)$ is partitioned into two

subsets such that the first one corresponds to subsets of N-1 and the other one corresponds to subsets of N which involve the index n. With the help of this partition of columns of R(S), the array R(S) can be partitioned to two arrays namely, R'(S) and R"(S). Thus R'(S) contains all the constraints of LP(N-1) and R"(S) contains all the constraints which are added to R'(S), to get all the constraints of LP(N).

Example 3.7 :

Let us consider example 3.5,

$$N = \{1,2,3,4,5,6,7,8\},\ N\text{-}1 = \{1,2,3,4,5,6,7\},$$

$$S = \{1,2,3,4\}.$$

R(S) is given in fig. 3.4.

The columns can be partitioned as follows;

Columns of R'(S) are

$$\phi,\ \{5\},\ \{6\},\ \{7\},\ \{5,6\},\ \{5,7\},\ \{6,7\},\{5,6,7\},$$ and

Columns of R"(S) are

$$\{8,\}\ \{5,8\},\{6,8\},\{7,8\},\{5,6,8\},\{5,7,8\},\{6,7,8\},\{5,6,7,8\}.$$

We define any block of R'(S) as

$$I^{k'}(S) = \{ T_k \cup I \mid T_k \subseteq \delta_k(S) \}$$

$$\text{for any } I \subseteq N\text{-}1 \setminus S, \tag{3.5.1}$$

and any block of R"(S) as

$$I^{k''}(S) = \{\{n\} \cup T_k \cup I \mid T_k \subseteq \delta_k(S)\} \tag{3.5.2}$$

Hence it is easy to see that there is a one-one correspondence between $R'(S)$ and $R''(S)$ because of the correspondence between (3.5.1) and (3.5.2). Thus $R''(S)$ can be viewed as a copy of $R'(S)$. This partition helps to extend any extreme point X^{n-1} of $F(N-1)$ to an extreme point X^n of $F(N)$. If X^{n-1} is an extreme point of first kind of $F(N-1)$ then $X^{n-1} \in V^*_{n-1}$ (As defined in sec. 2.3)

$\Longrightarrow$ $X^{n-1} = (a_1, a_2, \ldots, a_{n-1}, h)$ such that $a_i = 1$, $i \in S$.

We assume that $X^{n-1} = (1, 1, \ldots, 1, a_{s+1}, \ldots, a_{n-1}, h) \in V^*_{n-1}$.

The purpose is to obtain such extreme points X^n of $F(N)$ so that the common components of X^n and X^{n-1} have the same values. That is, to determine the possible values of a_n such that

$$(1, 1, \ldots, 1, a_{s+1}, a_{s+2}, \ldots, a_{n-1}, a_n, h) \in V^*_n$$

either is an extreme point of $F(N)$ or, gives rise to an extreme point of second kind of $F(N)$.

X^{n-1} is an extreme point of $F(N-1)$. Hence there are $\frac{n(n-1)}{2}$ linearly independent constraints which are in $R'(S)$. If X^n is an extreme point of $F(N)$ obtained by extending X^{n-1}, then at least $n = \left(\frac{n(n+1)}{2} - \frac{n(n-1)}{2}\right)$ linearly independent constraints in $R''(S)$ have to be made equalities. Theorem 3.9 gives the condition for existence of such n constraints in $R''(S)$.

Theorem 3.9 :

If for any a_n there exists at least one $I \subseteq (N-1)\setminus S$ such that $0 \leq \phi(I) \leq s-1$ where $\phi(I) = \theta(I) - a_n$, then there is an extreme point X^n of $F(N)$ such that

$$x_i^n = x_i^{n-1}\ ,\ x_{ij}^n = x_{ij}^{n-1}\ ,\ \text{for } 1 \leq i < j \leq n-1.$$

Proof : Let for some $I \subseteq (N-1)\setminus S$, $0 \leq \phi(I) \leq s-1$ the system of equations obtained by making $I^{\phi(I)''}(S)$, $I^{\phi(I)+1''}(S)$ equalities is

$$\sum_{i \in T_{\phi(I)}} x_i + \sum_{i \in I} x_i + x_n - \sum_{\substack{i<j\\ i,j \in T_{\phi(I)}}} x_{ij} - \sum_{\substack{i \in T_{\phi(I)}\\ j \in I}} x_{ij} - \sum_{\substack{i<j\\ i,j \in I}} x_{ij}$$

$$- \sum_{i \in T_{\phi(I)}} x_{in} - \sum_{i \in I} x_{in} = 1 \qquad (3.5.4)$$

and

$$\sum_{i \in T_{\phi(I)+1}} x_i + \sum_{i \in I} x_i + x_n - \sum_{\substack{i<j\\ i,j \in T_{\phi(I)+1}}} x_{ij} - \sum_{\substack{i \in T_{\phi(I)+1}\\ j \in I}} x_{ij}$$

$$- \sum_{\substack{i<j\\ i,j \in I}} x_{ij} - \sum_{i \in T_{\phi(I)+1}} x_{in} - \sum_{i \in I} x_{in} = 1 \qquad (3.5.5)$$

Let $T_{\phi(I)+1} = T_{\phi(I)} \cup \{k\}$ for $k \in S\setminus T_{\phi(I)}$. Then substituting the value of X^{n-1} in terms of a_i, h in (3.5.4) - (3.5.5) and substracting (3.5.4) from (3.5.5) we have,

$$\frac{2}{h+1} - \frac{2\ \phi(I)}{h(h+1)} - \frac{2 \sum_{i \in I} a_i}{h(h+1)} - x_{kn} = 0. \qquad (3.5.6)$$

Simplifying, we get $x_{kn} = \frac{2a_n}{h(h+1)}$ (3.5.7)

Since (3.5.4), (3.5.5) hold true for all $T_{\phi(I)} \in \delta_{\phi(I)}(S)$ and $T_{\phi(I)+1} \in \delta_{\phi(I)+1}(S)$, hence (3.5.7) holds for all $k \in S$.

Now using similar argument as theorem 3.2, substituting the values of (3.5.7) the whole set of constraints in a column of R''(S) can be reduced to either a single constraint or, a pair of constraints as follows.

For any column I of R''(S),

$$x_n - \sum_{i \in I} x_{in} \begin{cases} \leq RH\ \phi(I) = RH(\phi(I)+1), & \text{if } 0 \leq \phi(I) \leq s \\ \leq RH(0) & , \text{if } \phi(I) < 0 \\ \leq RH(s) & , \text{if } \phi(I) > s. \end{cases}$$

That is,

$$x_n - \sum_{i \in I} x_{in} \begin{cases} \leq \dfrac{a_n(a_n+1) + 2a_n\ \phi(I)}{h(h+1)} & , \text{if } 0 \leq \phi(I) \leq s \quad (3.5.8) \\ \leq \dfrac{a_n(a_n+1)}{h(h+1)} & , \text{if } \phi(I) < 0 \quad (3.5.9) \\ \leq \dfrac{a_n(a_n+1) + 2a_n\ s}{h(h+1)} & , \text{if } \phi(I) > s \quad (3.5.10) \end{cases}$$

Using the results of theorem 3.3 and 3.5 we conclude the following ;

(i) If there exist (n-s) number of subsets of $(N-1)\setminus S$ such that for all these subsets the number $\phi(I)$ satisfies $0 \leq \phi(I) \leq s$, and, when the set of inequalities (3.5.8), associated with these n-s subsets, are made equalities and if they give rise to a unique solution, then we get an extreme point of the first kind with values

$$x_n^n = \frac{(2h+1)\,a_n - a_n^2}{h(h+1)} \quad , \quad x_{in}^n = \frac{2a_i\,a_n}{h(h+1)} \quad i \in (N-1)$$

and rest of the coordinates are same as X^{n-1}.

(ii) Otherwise, the situation that at least for one subset I, $I \in (N-1)\setminus S$, $0 \leq \phi(I) \leq s-1$ ensures that by selecting some more constraints from (3.5.9) and (3.5.10) we can get $(n-s)$ subsets of $(N-1)\setminus S$ and when the corresponding constraints in (3.5.8) - (3.5.10) are made equalities they give rise to an extreme point of the second kind.

In fact the above two conclusions can be directly derived from theorem 3.3 and theorem 3.5.

Hence the proof is complete.

3.52 : Here we discuss the possible extension of any extreme point X^{n-1} of second kind of $F(N-1)$ to an extreme point of $F(N)$. Let $(1,1,\ldots,1,a_{s+1},\ldots,a_{n-1},h)$ be the point in $F(N-1)$ from which we generate the point $X^{n-1} \in$ Ext (2) as in section 3.4. Hence this point satisfies the conditions given in theorem 3.5. The aim is to find the values of a_n such that the point $(1,1,\ldots,1,a_{s+1},\ldots,a_{n-1},a_n,h) \in V^*$ either is an extreme point of $F(N)$ or, generates an extreme point of second kind of $F(N)$. The common coordinates of X^n, X^{n-1} which have same values are given as,

$$x_i^n = x_i^{n-1} = \frac{2}{h+1} \ , \ i \in S$$

$$x_{ij}^n = x_{ij}^{n-1} = \frac{2}{h(h+1)} \ , \ i < j, \ i,j \in S,$$

where s is defined in theorem 3.5. In view of theorem 3.3 and theorem 3.5, like theorem 3.9, conditions a_n in terms h, a_i, $i \in N-1\setminus S$ can be derived.

Thus such a point X^n can be viewed as the extension of the extreme point X^s of F(S) to F(N), where X^s is

$$x_i^s = \frac{2}{h+1}\ ,\ x_{ij}^s = \frac{2}{h(h+1)}\ ,\ 1 \leq i < j \leq s.$$

Clearly X^s is an extreme point of first kind of F(S).

3.53 : So far we have shown that there are some extreme points of F(N) which are obtained as a result of extensions of Ext (1) of F(N-t), $t > 0$. Here we give an exceptional case. We give an example of one extreme point of F(N) which cannot be obtained as a result of such extensions.

Let $X^n \in F(N)$ be defined as

$$x_i^n = \frac{2}{n-1}\ ,\ x_{ij} = \frac{2}{(n-1)(n-2)}\ ,\quad 1 \leq i < j \leq n.$$

Clearly $X^n \in$ Ext (1) and $X^n \in V^*$.

In order that X^n is an extension of any extreme point of F(S), $S \subseteq N$, it is necessary that the point X^s defined as

$$x_i^s = x_i^n,\ x_{ij}^s = x_{ij}^n,\quad i < j,\ i,j \in S$$

is an extreme point of F(S).

But from theorem 3.1, the point

$x_i = \frac{2}{h+1}\ ,\ x_{ij} = \frac{2}{h(h+1)}\ ,\ i < j,\ i,j \in S$ is an extreme point of F(S) for $0 \leq h \leq s-2$ and hence X^s is not an extreme point of F(S).

We believe that except for the above particular case all the extreme points of $F(N)$ can be generated by the method given in 3.51 and 3.52 from the extreme points of $F(N-t)$, $t > 0$ and all these observations are summed up in the following conjecture.

Conjecture 3.1

For any extreme point x^n of $F(N)$ there exists a subset $S \subsetneq N$ such that $2 \leq s \leq n$ and

$$x_i^n = \frac{2}{s-1}, \quad x_{ij}^n = \frac{2}{(s-2)(s-1)}, \quad \forall\, i < j,\ i,j \in S.$$

3.6 *Extreme points in V_i, $1 \leq i \leq 8$.*

To strengthen our belief we show that the classes of points V_4, V_5, V_6, V_7, V_8, which are not in Ext (1), Ext (2), are not extreme points of $F(N)$

Theorem 3.10

The vertex points V_4, V_5, V_6, V_7, V_8 defined in (1.5.4)-(1.5.8) are not extreme points of the convex polyhedron of $F(N)$.

Proof: In theorem 2.1 part II it is proved that the members of V_4, V_5, V_6, V_7 can be expressed as the convex combination of V^*. So it only remains to show that none of the points of V_8 is extreme points of $F(N)$. From definition 1.1, any extreme point of $F(N)$ is a solution to some linear system of $\frac{n(n+1)}{2}$ equations which have rational coefficients namely 0,1.

Hence any point which is solution to this system can not have irrational coordinate values. Any point $X \in V_8$ which is defined by (1.5.8) from the set of numbers $b_1, b_2 \ldots b_n$ out of which at least one b_i is irrational then from (1.5.8) we have

$$x_i = 2b_i - b_i^2 \qquad x_{ij} = 2b_i b_j$$

at least one coordinate of X as irrational value, and hence such X can not be an extreme point of the system F(N). Thus it is sufficient to check only those $X \in V_8$ which are defined by the set of rational numbers b_i, $1 \leq i \leq n$.

Any such numbers are of the form

$$b_i = \frac{p_i}{q_i} = \frac{a_i}{h}$$

where p_i, q_i are relatively prime, $q_i > 0$ and h defined as $h = \text{LCM}\ [q_1, q_2 \ldots q_n]$.

We shall show that when $h \neq 1$, the associated $X \in V_8$ can be expressed as the convex combination of the points of V^* and if $h = 1$, then the point $X \in V_8$ is in the convex combination of two feasible points of F(N).

Case i If $h \neq 1$, we determine two points of V^* as follows

$$X' = (a_1, a_2, \ldots, a_n, h) \in V^*$$

and

$$X'' = (a_1, a_2, \ldots, a_n, h-1) \in V^*$$

Hence

$$x_i' = \frac{(2h+1)\ a_i - a_i^2}{h(h+1)}, \quad x_{ij}' = \frac{2a_i a_j}{h(h+1)}$$

$$x_i'' = \frac{(2h-1)a_i - a_i^2}{h(h-1)}, \quad x_{ij}'' = \frac{2a_i a_j}{h(h-1)}$$

Now X is in $C[X', X'']$ as follows

$$X = \frac{h+1}{2h} X' + \frac{h-1}{2h} X''$$

To see this

$$\frac{2h+1}{2h} x_i' + \frac{h-1}{2h} x_i'' = \frac{2h\ a_i - a_i^2}{h^2}$$

$$= 2\frac{a_i}{h} - \frac{a_i^2}{h^2} = 2b_i - b_i^2 = x_i$$

$$\text{and } \frac{h+1}{2h} x_{ij}' + \frac{h-1}{2h} x_{ij}'' = \frac{2a_i a_j}{h^2} = 2\ b_i b_j = x_{ij}$$

Thus for $X \in V_8$, there are two point $X', X'' \in V^*$ such that $X \in C[X', X'']$. Hence such X can not be extreme point of $F(N)$.

Case ii If $h = 1$ i.e., if all b_i $1 \leq i \leq n$ are integers for the point $X \in V_8$ under consideration, in this case case X' can not be defined. For such cases it is shown that for any such $X \in V_8$ there exists a point $X' \in V^*$ such that the point

$$Y = X' + \lambda(X - X'), \text{ for any } \lambda > 0$$

is a feasible point of $F(N)$. Hence $X \in C[X', Y]$. Since $h = 1 \implies b_i = a_i$, $1 \leq i \leq n$, $X' = (a,1) \in V^*$

$$x_i = \frac{(2h+1)a_i - a_i^2}{h(h+1)} = \frac{3b_i - b_i^2}{2}$$

$$x_{ij} = \frac{2a_i a_j}{h(h+1)} = b_i b_j$$

Now

$$y_i = x_i' + \lambda(x_i - x_i') = x_i' + \lambda\left(\frac{b_i - b_i^2}{2}\right)$$

$$y_{ij} = x_{ij}' + \lambda(x_{ij} - x_{ij}') = x_{ij}' + \lambda(b_i b_j)$$

For any arbitrary constraint $J \subseteq N$

$$\sum_{i \in J} y_i - \sum_{\substack{i < j \\ i,j \in J}} y_{ij}$$

$$= \sum_{i \in J} x_i' - \sum_{\substack{i < j \\ i,j \in J}} x_{ij}' + \lambda\left\{ \sum_{i \in J} \frac{b_i - b_i^2}{2} - \sum_{\substack{i < j \\ i,j \in J}} b_i b_j \right\}$$

$$= \sum_{i \in J} x_i' - \sum_{\substack{i < j \\ i,j \in J}} x_{ij}' + \frac{\lambda}{2}\left\{ \sum_{i \in J} b_i - \left(\sum_{i \in J} b_i\right)^2 \right\}$$

Since for any integers b_i, $\Sigma\, b_i - (\Sigma\, b_i)^2 \leq 0$. Hence, for $\lambda > 0$

$$\sum_{i \in J} y_i - \sum_{\substack{i < j \\ i,j \in J}} y_{ij} \leq \sum_{i \in J} x_i' - \sum_{\substack{i < j \\ i,j \in J}} x_{ij}' \leq 1$$

for X' is a feasible point

$\Longrightarrow$ Y is a feasible point

$\Longrightarrow$ X is the convex combination of two feasible point

$\Longrightarrow$ $X \in V_8$ is not an extreme point.

This completes the proof.

3.7 Adjacency structure of extreme points

In the section, an attempt is made to study some aspects of the adjacency structure of the extreme points

Definition 3.3: Adjacent extreme points

Two extreme points X', X'' of $F(N)$ are said to be adjacent if there exist $\alpha'(N) \subseteq \mu'(N)$, $\alpha''(N) \subseteq \mu''(N)$ where $\mu'(N)$, $\mu''(N)$ are the set of constraints which are equalities at X', X'' respectively, satisfying the following relations.

$$|\alpha'(N)| = |\alpha''(N)| = \frac{n(n+1)}{2}$$

$$r_2(\alpha'(N)) = r_2(\alpha''(N)) = \frac{n(n+1)}{2}$$

and $$|\alpha'(N) \cap \alpha''(N)| = \frac{n(n+1)}{2} - 1$$

We discuss here the adjacency among a pair of distinct extreme points in Ext (1) satisfying the following conditions.

Let $X' = (a', h) \in V^*$, $X'' = (a'', h) \in V^*$. Let there be a subset S, $h \geq s-2$ such that

$$a_i' = a_i'' = 1, \quad \forall \ i \in S.$$

and $$a_i'' = a_i' \pm \delta_i \ \ i \in N \setminus S, \ \text{where } \delta_i \text{ is } 0 \text{ or, } 1.$$

We here assume that there exists at least one $i \in N \setminus S$ such that $a_i'' \neq a_i'$. Since X', X'' are extreme points of first kind, it follows that

(i) There exist collections of subsets of $N \setminus S$, i.e. $\alpha'(N \setminus S)$, $\alpha''(N \setminus S)$ such that

$$r_2(\alpha'(N \setminus S)) = r_2(\alpha''(N \setminus S)) = |\alpha'(N \setminus S)| = |\alpha''(N \setminus S)| = \frac{(n-s)(n-s+1)}{2}$$

and $$-1 \leq h - \sum_{i\in I} a_i' \leq s \quad \forall\, I \in \alpha'(N\setminus S)$$

$$-1 \leq h - \sum_{i\in I} a_i'' \leq s \quad \forall\, I \in \alpha''(N\setminus S)$$

(ii) There exists $\beta'(N\setminus S) \subseteq \alpha'(N\setminus S)$, $\beta''(N\setminus S) \subseteq \alpha''(N\setminus S)$

such that $$r_1(\beta'(N\setminus S)) = |\beta'(N\setminus S)|$$

$$= r_1(\beta''(N\setminus S)) = |\beta''(N\setminus S)| = (n-s)$$

and $$0 \leq h - \sum_{i\in I} a_i' \leq s-1 \quad \forall\, I \in \beta'(N\setminus S)$$

$$0 \leq h - \sum_{i\in I} a_i'' \leq s-1 \quad \forall\, I \in \beta''(N\setminus S).$$

Now let us define the following collections of subsets of $N\setminus S$.

$$\gamma_0 = \{I \subseteq N\setminus S : |\sum_{i\in I} a_i' - \sum_{i\in I} a_i''| = 0\}$$

$$\gamma_1 = \{I \subseteq N\setminus S : |\sum_{i\in I} a_i' - \underset{i\in I}{} a_i''| \leq 1\}$$

$$\gamma_> = \{I \subseteq N\setminus S : |\sum_{i\in I} a_i' - \sum_{i\in I} a_i''| > 1\}.$$

Obviously, $\gamma_0 \subseteq \gamma_1$ and $\gamma_1 \cup \gamma_> = N\setminus S$.

In order to find the condition that X', X'' are adjacent in $F(N)$ we first prove lemma 3.6 which is useful for the purpose.

<u>Lemma 3.6</u>

$$r_1(\gamma_0 \cap \beta'(N\setminus S)) \leq n-s-1.$$

<u>Proof</u>: Clearly $r_1(\gamma_0 \cap \beta'(N\setminus S)) \leq n-s$. It suffices to show that

$$r_1(\gamma_0 \cap \beta'(N\setminus S)) \neq n-s.$$

We know that $r_1(\beta'(N\setminus S)) = n-s$ (3.7.1)

$$r_1(\gamma_o \cap \beta'(N\setminus S)) = n-s$$

$$\Longrightarrow \gamma_o \cap \beta'(N\setminus S) = \beta'(N\setminus S).$$

Hence

$$\sum_{i\in I} a_i' - \sum_{i\in I} a_i'' = 0 \quad \forall\ I \in \beta'(N\setminus S) \tag{3.7.2}$$

From (3.7.1) clearly the system of equations(3.7.2) is equivalent to the following

$$a_i' = a_i'' \quad \forall\ i\in N\setminus S.$$

Hence is a contradiction, for we have assumed that at least for one $i\in N\setminus S$, $a_i' \neq a_i''$.

Hence $r_1(\gamma_o \cap \beta'(N\setminus S)) \neq n-s$ and the proof is complete.

<u>Theorem 3.11</u>

If $r_1(\gamma_o \cap \beta'(N\setminus S)) = n-s-1$ (3.7.3)

and $r_2(\gamma_1 \cap \alpha'(N\setminus S)) = \frac{(n-s)(n-s+1)}{2}$ (3.7.4)

then the points X', X'' are adjacent in $F(N)$.

<u>Proof</u>: Since $|\alpha'(N\setminus S)| = r_2(\alpha(N\setminus S)) = \frac{(n-s)(n-s+1)}{2}$,

(3.7.4) implies that

$$\gamma_1 \cap \alpha'(N\setminus S) = \alpha'(N\setminus S). \tag{3.7.5}$$

Again from (3.7.3) we directly get that

$$|\gamma_o \cap \beta'(N\setminus S)| = n-s-1$$

$$\Longrightarrow \gamma_o \cap \beta'(N\setminus S) = \beta'(N\setminus S) | \hat{I} \text{, for some } \hat{I}\in\beta'(N\setminus S)$$

Clearly such $\hat{I} \notin \gamma_0$ but since $\hat{I} \in \beta'(N\backslash S) \subseteq \alpha'(N\backslash S)$

$$\Longrightarrow \hat{I} \in \gamma_1 \text{, from (3.7.5)}$$

Hence $$\left| \sum_{i\in\hat{I}} a_i' - \sum_{i\in\hat{I}} a_i'' \right| = 1$$

$$\Longrightarrow \left|\theta'(\hat{I}) - \theta''(\hat{I})\right| = 1$$

where $\theta'(\hat{I}) = h - \sum_{i\in\hat{I}} a_i'$ and $\theta''(\hat{I}) = h - \sum_{i\in\hat{I}} a_i''$.

Let us assume that $\theta'(\hat{I}) - \theta''(\hat{I}) = 1$

$$\Longrightarrow \theta''(\hat{I}) = \theta'(\hat{I}) - 1 \tag{3.7.6}$$

From sections 3.31–3.34, we know that the constraints $\hat{I}^{\theta'(\hat{I})}, \hat{I}^{\theta'(\hat{I})+1}$ are equalities at the point X' and similarly $\hat{I}^{\theta''(\hat{I})}, \hat{I}^{\theta''(\hat{I})+1}$ are equalities. From (3.7.6) we have $\hat{I}^{\theta''(\hat{I})+1} = \hat{I}^{\theta'(\hat{I})}$.

The set of constraints which are equalities at X' are given by, interms of blocks of $R(S)$.

$$\phi^h,\ \phi^{h+1},\ I^{\theta'(I)},\ I^{\theta'(I)+1},\ \forall\ I \in \alpha'(N\backslash S) \tag{3.7.7}$$

Similarly the set of constraints which are equalities at X'' are

$$\phi^h,\ \phi^{h+1},\ I^{\theta''(I)},\ I^{\theta''(I)+1},\ \forall\ I \in \alpha''(N\backslash S) \tag{3.7.8}$$

Using the relations (3.7.3)–(3.7.8) we can write some constraints which are equalities at both X', X'' as

$$\phi^h,\ \phi^{h+1},\ I^{\theta'(I)},\ I^{\theta'(I)+1},\ \forall\ I \in \beta'(N\backslash S)\backslash\hat{I}$$

$$\hat{I}^{\theta(\hat{I})}$$

$$I^{\theta'(I)},\ I^{\theta'(I)+1},\ \forall\ I \in \alpha'(N\backslash S) \cap \gamma_0$$

$I^{\theta'(I)}$, for all $I \in \alpha'(N\backslash S)$ such that $\sum_{i\in I} a_i' - \sum_{i\in I} a_i'' = -1$

$I^{\theta'(I)+1}$, for all $I \in \alpha'(N\backslash S)$ such that $\sum_{i\in I} a_i' - \sum_{i\in I} a_i'' = +1$.

From the definition 3.2, it suffices to show that the above collection of subsets has rank $r_2 = \frac{n(n+1)}{2} - 1$ and when either $\hat{I}^{\theta'(\hat{I})+1}$ or, $\hat{I}^{\theta''(\hat{I})}$ is added the rank r_2 is $\frac{n(n+1)}{2}$. Repeating the step by step argument used in section 3.3 the above facts can be proved. To avoid unnecessary repetition, we omit the rest of the proof.

The following example illustrates the above result.

Example 3.7

Let us take the same example as example 3.3. Let X' be defined as

$a_5=1$, $a_6=1$, $a_7=2$, $a_8=-1$, $h=2$ (Fig. 1.4)

and X'' be defined as

$a_5=1$, $a_6=1$, $a_7=1$, $a_8=0$, $h=2$

$\alpha'(N\backslash S) = \{\{i\}, \{i,j\} \mid \forall\ i < j,\ i,j \in N\backslash S\}$

and $\alpha''(N\backslash S) = \alpha'(N\backslash S)$

$\beta'(N\backslash S) = \{\{5\}, \{6\}, \{7\}, \{78\}\}$

$\gamma_o = \{\{5\}, \{6\}, \{78\}\}$

$\gamma_1 = N\backslash S$ and $\gamma_> = \phi$.

$\gamma_1 \cap \alpha'(N\backslash S) = \alpha'(N\backslash S)$ so $r_2(\alpha'(N\backslash S)) = \frac{(n-s)(n-s+1)}{2}$

$r_1\ (\beta'(N\backslash S)) = n-s$

and $r_1\ (\beta'(N\backslash S) \cap \gamma_o) = n-s-1$

Hence X', X'' are adjacent.

Like example 3.3, we can choose suitably $\frac{n(n+1)}{2}$ constraints for each X',X'' seperately and show that they have $\frac{n(n+1)}{2}-1$ constraints as equalities in common. The set of constraints which will define X' is

{12} , {13} , {14} , {23} , {24} , {34} , {123} , {124} , {134} , {234} ,{15} , {25} , {35} , {45} , {125} , {16} , {26} , {36} , {46} , {126} , {7} , {17} , {27} , {37} , {47} , {178} ,{ 278} , { 378 } , {478 } , {1278} , {56} , { 57} , {1258} , {67} , {1268} , {1238} .

The set of constraints which will define X'' is

{12} , {13} , {14} ,{ 23} ,{ 24} , {34} ,{ 123} , {124} , {134} , {234} , {15} , {25} , {35} , {45} , {125} , {16} , {26} , {36} , {46} ,{ 126} , {17} , {27} , {37} , {47} , {127} , {178} , {278} , {378} , {478} , { 1278} , {56} , {57} , {1258} , {67} , {1268} , {1238} .

We can see that except the constraint {7} rest of constraints which define X', also define X'' together with the constraint {1,2,7} . Hence by definition 3.3,

X', X'' are adjacent.

Remark: In order to show the adjecency among any given pair of points suitable $\beta'(N\backslash S)$ and $\alpha'(N\backslash S)$ should be chosen. For instance, if we take

$$X''=(a_1=1,\ a_2=1,\ a_3=1,\ a_4=1,\ a_5=1,\ a_6=1,\ a_7=1,\ a_8=-1, h=2).$$

Then by choosing a different $\beta'(N\backslash S)$ it can be shown that X' and X'' are adjacent in F(N).

Chapter IV

SPECIAL CASES

4.1 *Introduction*

This chapter deals with some special cases for which the best Bonferroni lower bound can be computed.

Section 4.2 gives some useful results concerning the special cases discussed in section 4.3. Section 4.4 deals with the condition under which a specific extreme point will be optimal.

4.2 *Elementary results*

This section presents some results concerning the constraints that are equalities at the optimal point of LP(N). Though all the results are not directly used for computing the bounds some of them are important in this respect.

<u>Lemma 4.1</u> :

If for some i,

$$P_i - \sum_{j \neq i} P_{ij} \geq 0,$$

then $x_i = 1$ at the optimal point of LP(N).

<u>Proof</u> : We prove the lemma by contradiction. Let us consider that X^* be the optimal point and for some i, $1 \leq i \leq n$,

$$x_i^* < 1$$

and $$P_i - \sum_{j \neq i} P_{ij} \geq 0.$$

We show that there is another feasible point $\hat{X}$ such that $\hat{x}_i = 1$ and the value of the objective function at $\hat{X}$ is not worse than that at X^*.

Let $d = 1-x_i^*$. Then clearly $d > 0$. Define the point $\hat{X}$ as follows,

$$\hat{x}_j = \begin{cases} x_j^* & , j \neq i \\ x_j^*+d, & j = i \end{cases}, \quad \hat{x}_{jk} = \begin{cases} x_{jk}^* & , j \neq i,\ k \neq i \\ x_{ij}^*+d, & i = k,\ j \neq i. \end{cases}$$

We claim that $\hat{X} \in F(N)$. Let us consider any constraint $J_r \subseteq N$ such that $i \in J_r$. Then

$$\sum_{k \in J_r} \hat{x}_k - \sum_{\substack{k < j \\ k,j \in J_r}} \hat{x}_{kj}$$

$$= \sum_{k \in J_r} x_k^* - \sum_{\substack{k < j \\ k,j \in J_r}} x_{kj}^* + d - (r-1)d \qquad (4.2.1)$$

If $r > 1$, then $d - (r-1)d < 0$.

$$\Longrightarrow \sum_{k \in J_r} \hat{x}_k - \sum_{\substack{k < j \\ k,j \in J_r}} \hat{x}_{kj}$$

$$< \sum_{k \in J_r} x_k^* - \sum_{\substack{k < j \\ k,j \in J_r}} x_{kj}^* \leq 1 \quad \because X^* \in F(N) \qquad (4.2.2)$$

If $r = 1$, then $J_r = \{i\}$, $\hat{x}_i = x_i^* + d = 1 \qquad (4.2.3)$

Since the constraints which do not involve the index i are not affected by the change made from X^* to $\hat{X}$, (4.2.2) - (4.2.3) show that the point $\hat{X} \in F(N)$.

The difference in the value of the objective function at $\hat{X}$ and X^* is given by

$$\bar{P}^T\hat{X} - \bar{P}^T X^* = d(P_i - \sum_{j \neq i} P_{ij}) \geq 0$$

Hence $\bar{P}^T\hat{X} \geq \bar{P}^T X^*$.

This completes the proof.

Lemma 4.2 :

If for some subset $S \subseteq N$ and $i \notin S$, there is an integer k such that

$$P_i - \sum_{j \neq i} P_{ij} + \frac{k-1}{k} \sum_{j \in S} P_{ij} \geq 0 \qquad (4.2.4)$$

then at least one of the constraints from

$$\bigcup_{t=0}^{k-1} (\delta_t(S) \cup \{i\})$$

is equality at the optimal point.

Proof : Let X^* be the optimal point. Assume that none of the constraints of

$\bigcup_{t=0}^{k-1} (\delta_t(S) \cup \{i\})$ is equality at X^*. Define

a point $\hat{X}$ such that for some $d > 0$,

$$\hat{x}_i = x_i^* + d\ , \text{ for } j \not\in S,\ j \neq i$$

$$\hat{x}_{ij} = x_{ij}^* + d, \text{for } j \not\in S,\ j \neq i$$

$$\hat{x}_{ij} = x_{ij}^* + \frac{1}{k}\, d,\ j \in S$$

$$\left.\begin{aligned} \hat{x}_{jq} &= x_{jq}^*, \\ \hat{x}_j &= x_j^* \end{aligned}\right\} \text{ otherwise.}$$

Any subset $J \subseteq N$, $i \in J$ can be written as

$$J = \{i\} \cup T_t \cup T_p,\ T_t \in S,\ T_p \in (N \backslash S) \setminus \{i\}.$$

Then the constraint associated with J can be written for $\hat{X}$ as follows,

$$\sum_{j \in J} \hat{x}_j - \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \in J}} \hat{x}_{j_1 j_2}$$

$$= \sum_{j \in J} x_j^* - \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \in J}} x_{j_1 j_2}^* + d\left(1 - p - \frac{t}{k}\right) \qquad (4.2.5)$$

For any $p \geq 1$, $d > 0$, we have from (4.2.5),

$$\sum_{j \in J} \hat{x}_j - \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \in J}} \hat{x}_{j_1 j_2} \leq 1, \quad \because X^* \in F(N).$$

Thus constraints which do not involve the index i are automatically satisfied by the point $\hat{X}$. The constraints which are affected by changing the solution from X^* to $\hat{X}$ are

$$\bigcup_{t=0}^{k-1} (\delta_t(S) \cup \{i\}).$$

In order that the point $\hat{X}$ to satisfy above constraints the value of d is restricted as follows,

$$d = \underset{J \in \bigcup_{t=0}^{k-1} (\delta_t(S) \cup \{i\})}{\text{Min}} \left[1 - \sum_{j \in J} x_j^* + \sum_{\substack{j_1, j_2 \\ j_1 < j_2 \in J}} x_{j_1 j_2}^* \right] \tag{4.2.6}$$

For any $d > 0$ satisfying (4.2.6) the value of the objective function increases from X^* to $\hat{X}$. Moreover (4.2.7) ensures that one of the constraints of

$$\bigcup_{t=0}^{k-1} (\delta_t(S) \cup \{i\}) \text{ is equality.}$$

Lemma 4.3 :

If for some i, $P_i - \frac{1}{n-2} \sum_{j \neq i} P_{ij} < 0$, then the constraint associated with J_n is an equality at the optimal point.

Proof : We follow the same method of proof as in lemma 4.2. Let X^* be optimal point and $P_i - \frac{1}{n-2} \sum_{j \neq i} P_{ij} < 0$. Define $\hat{X}$ as,

$$\hat{x}_i = x_i^* - d,\ \hat{x}_{ij} = x_{ij}^* - \frac{1}{n-2} d,\ j \neq i \text{ for some } d > 0$$

and $\hat{x}_k = x_k^*$, $\hat{x}_{jk} = x_{jk}^*$, $j \neq i$, $k \neq i$.

It is easy to see that as d increases the value of the objective function increases and the point $\hat{X}$ remains feasible as

long as the constraint J_n is satisfied. Hence the maximum increase in the objective function is possible by the maximum possible value of d, i.e. when $\hat{X}$ satisfies the constraint J_n as equality.

This completes the proof.

4.3 *Special cases*

In this section we give some of the special cases for which the Bonferroni lower bound can be calculated by direct inspection.

Theorem 4.1 deals with one of the simplest cases which is ignored in earlier works.

Theorem 4.1 :

If $P_i - \sum_{j \neq i} P_{ij} \geq 0, \; \forall \; i$, then the optimal point of LP(N) is given by

$$x_i = x_{ij} = 1, \quad 1 \leq i < j \leq n$$

and the exact lower bound is

$$\sum_{i=1}^{n} P_i - \sum_{1 \leq i < j \leq n} P_{ij}$$

Proof : From lemma 4.1 we know that $x_i = 1$ for all i at the optimal point. Substituting the values of x_i in LP(N), we have

$$\text{Max} \sum_{i=1}^{n} P_i - \sum_{1 \leq i < j \leq n} P_{ij} x_{ij}$$

$$\text{such that } r - \sum_{\substack{i < j \\ i,j \in J_r}} x_{ij} \leq 1 \quad \forall \; J_r \subseteq N, \; 1 \leq r \leq n.$$

The solution to the above linear program can be obtained directly as follows. Since $P_{ij} \geq 0$, $1 \leq i < j \leq n$ and since $x_{ij} \geq 1$, the optimal point is given by

$$x_{ij} = 1, \; 1 \leq i < j \leq n.$$

This completes the proof.

Theorem 4.2 gives a generalization of the result by Gallot [17], and Kounias [29]. They show that for $n = 2$, if $P_1 = P_2 = P_{12}$ the lower bound $P^T Q^- P$ is exact. We give a wider class of points for which this bound is exact.

<u>Theorem 4.2</u> :

If $P_{ij} = \mathrm{Min}\,(P_i, P_j)$, $1 \leq i < j \leq n$ then $P^T Q^- P$ is the optimal value of LP(N).

<u>Proof</u> : We shall show that there is a feasible solution Y^o of DP(N) such that the value of the dual objective function at Y^o is $P^T Q^- P$. Since there is a feasible point in LP(N) such that the objective function at this point has the value $P^T Q^- P$, from the theory of duality we have $P^T Q^- P$ as the optimal value of dual pair of linear programs.

Let $P_1 \geq P_2 \geq \ldots \geq P_n$.

Consider the subsets

$$J^1 = \{1\}, \quad J^2 = \{1,2\}, \quad J^3 = \{1,2,3\} \ldots J^n = \{1,2,\ldots,n\}$$

We define the dual variable Y^o as

$$y^o(J^\ell) = P_\ell - P_{\ell+1} \qquad 1 \leq \ell \leq n-1$$

$$y^o(J^n) = P_n \quad \text{and} \quad y^o(J) = 0 \quad \forall\, J \neq J^{\ell}, \; 1 \leq \ell \leq n$$

It is easy to check that

$$\sum_{J \subseteq N} y^o(J)\, d(J) = \bar{P} \quad \text{and} \quad y^o(J) \geq 0, \quad \forall\, J \subseteq N.$$

Hence Y^o is a feasible point of DP(N).

The value of the objective function at Y^o is given by

$$\sum_{k=1}^{n} y^o(J^k) = P_1.$$

It is also seen that $Q^{-}P$ is given by $[1,0,0,\ldots,0]$. Hence $P^T Q^{-} P = P_1$.

This completes the proof.

Remark : In order to compute the best bound for such cases the following relation can be made use of

$$P^T Q^{-} P = \max_{1 \leq i \leq n} P_i.$$

In theorem 4.3 we consider the case when all P_i are same and all the P_{ij} are also same. This case is also treated in [13]; [29], [31] , [47]. Here we derive the result using duality theory.

Theorem 4.3 :

If $P_i = P_1$, $P_{ij} = P_{12}$, $1 \leq i < j \leq n$, then the best lower bound obtained is

$$\frac{2n}{k+1} P_1 - \frac{n(n-1)}{k(k+1)} P_{12} \qquad (4.3.1)$$

where $(k-1)$ is the greatest integer less than $\dfrac{(n-1)\,P_{12}}{P_1}$.

Proof : The proof is similar as that of theorem 4.2. It is shown that the value (4.3.1) is taken on by the primal and dual objective function at their respective feasible point.

Define $X \in F(N)$ as,

$$x_i = \frac{2}{k+1}\ , \quad x_{ij} = \frac{2}{k(k+1)}\ .$$

The dual feasible solution Y^o is defined as

$$y^o(J_k) = r_1, \quad y^o(J_{k+1}) = r_2$$

for all $J_k \in \delta_k(N)$

$J_{k+1} \in \delta_{k+1}(N)$, where,

$$r_1 = \frac{n-1}{{}^{n-1}C_{k-1}} \left[\frac{kP_1}{n-1} - P_{12}\right]$$

$$r_2 = \frac{n-1}{{}^{n-1}C_k} \left[P_{12} - \frac{k-1}{n-1} P_1\right]$$

Clearly $r_1 \geq 0$, $r_2 \geq 0$. It is easy to check that

$$r_1 \sum_{J_k \in \delta_k(N)} d(J_k) + r_2 \sum_{J_{k+1} \in \delta_{k+1}(N)} d(J_{k+1}) = \bar{P}.$$

Hence Y^o is a feasible point of DP(N).

The value of the objective function at Y^o is

$$^nC_k\, r_1 + {}^nC_{k-1}\, r_2 \tag{4.3.2}$$

Simplifying (4.3.2) we have (4.3.1).

The following theorem is a generalisation of earlier theorem 4.3. Though the result of theorem 4.3 is also obtained by Kounias and Marin [31], such generalisation becomes possible only when it is proved by the help of duality theory.

Theorem 4.4 :

If $P_i = P_1$, $P_{ij} = P_{12}$, $P_{in} = P_{1n}$ for all $1 \leq i < j \leq n$, then the best lower bound is given by

$$\frac{2(n-1)}{k+1} P_1 - \frac{(n-1)(n-2)}{k(k+1)} P_{12} + \left(1 - \frac{t(t+1)}{k(k+1)}\right) P_n - \frac{2(k-t)(n-1)}{k(k+1)} P_{1n} \quad (4.3.3)$$

where $t-1$, $k-1$ are the greatest integers less than

$$\frac{(n-1)\,P_{1n}}{P_n}, \quad \frac{\left[P_{12} - \frac{t-1}{n-1}\left(2P_{1n} - \frac{t}{n} P_n\right)\right]}{P_1 - P_{1n}}, \text{ respectively.}$$

Proof : The dual feasible solution Y^o can be defined as,

$$y^o(T_k) = r_1, \quad y^o(T_{k+1}) = r_2, \quad y^o(T_{t+1} \cup \{n\}) = r_3,$$

$$y^o(T_{t+1} \cup \{n\}) = r_4,$$

for all $T_k \in \delta_k(N-1)$, $T_{k+1} \in \delta_{k+1}(N-1)$,

$T_t \in \delta_t(N-1)$, $T_{t+1} \in \delta_{t+1}(N-1)$,

and, $y^o(J) = 0$ for the remaining subsets $J \subseteq N$. The values of r_1, r_2, r_3, r_4 are defined as

$$r_1 = \frac{1}{{}^{n-2}C_{k-1}} \left[k(P_1 - P_{1n}) - P_{12} + \frac{t-1}{n-1}\left(2P_{1n} - \frac{t}{n} P_n\right) \right]$$

$$r_2 = \frac{1}{{}^{n-2}C_k} \left[P_{12} - \frac{t-1}{n-1}\left(2P_{1n} - \frac{t}{n} P_n\right) - (k-1)\, P_1 - P_{1n} \right]$$

$$r_3 = \frac{n-1}{{}^{n-1}C_{t-1}} \left[t \frac{P_n}{n-1} - P_{1n} \right]$$

$$r_4 = \frac{n-1}{{}^{n-1}C_t} \left[P_{1n} - \frac{t-1}{n-1} P_n \right] .$$

It is easy to see that Y^o is a feasible basic solution of DP(N). It also provides an optimal point of DP(N), because the value at this point is same as the value of the objective function of LP(N) at the point,

$$x_i = \frac{2}{k+1} , \quad x_{ij} = \frac{2}{k(k+1)} , \quad 1 \leq i < j \leq n-1$$

$$x_n = 1 - \frac{(t+1)(t+2)}{k(k+1)} , \quad x_{in} = \frac{2(k-t)}{k(k+1)} , \quad 1 \leq i \leq n.$$

Remark : Such cases as discussed in theorem 4.3, 4.4, can be extended to more general cases, but the simplicity with which the best lower bound is calculated is not retained. Hence, further generalization will not lead to simple computational procedure and hence are not of much use. The possible extension of theorem 4.4 can be thought as,

for some $S \subseteq N$,

$$P_i = P_1, \; P_{ij} = P_{12}, \; P_{it} = P_{in}, \; i < j, \; i,j \in S, \; t \notin S$$

$$P_t = P_n, \; P_{tk} = P_{n-1n}, \quad t < k, \; t,k \notin S.$$

For such problems, the condition of optimality can be generalised following theorem 4.4. This problem is treated by Kounias [30]

4.4 *Optimality of specific extreme points*

In this section we discuss conditions under which certain special kind of extreme points are optimal. In order to determine such condition, we make use of the following results of duality theory.

If $X \in F(N)$ is an extreme point of $F(N)$, then there are $\frac{n(n+1)}{2}$ independent vectors $d(J)$ such that $d(J)\, X = 1$. In DP(N), these vectors $d(J)$ correspond to columns of A^T associated with the dual variables $y(J)$. Let B be the square submatrix of A^T obtained by picking up these $\frac{n(n+1)}{2}$ columns of A^T. If B is nonsingular then B constitutes a basis of DP(N). From the well known theorem of duality theory, if $B^{-1}\bar{P} \geq 0$, then X is the optimal point of LP(N). To show that B is nonsingular and $B^{-1}\bar{P} \geq 0$, we adopt the algorithm suggested by Murty [39]. Since the rows of B are coefficients of components of X in LP(N), we denote the rows of B as R_i corresponding to component x_i of X and R_{ij} as corresponding to component x_{ij} of X.

The matrix B is written in a tableau and in the course of the algorithm the rows of the tableau are altered such that at every step each row in the tableau is a nonzero linear combination of the rows of the initial tableau. If at a particular stage all the entries of any row or column are zero, then the matrix is singular because the zero vector is shown as the linear combination of the rows of B. Otherwise the algorithm stops when by successive row operation the tableau contains the

identity matrix. If any elementary row operation is represented algebraically as $R_i + kR_j$ which means that k times of row R_j is added to row R_i, then the rows $\bar{R}_i, \bar{R}_{ij}$ of the final tableau obtained after successive row operations, can be represented as linear algebraic function of R_i, R_{ij}.

Let us first consider the extreme point x', where

$$x_i' = \frac{2}{n-1}\ ,\ x_{ij}'' = \frac{2}{(n-1)(n-2)} \qquad 1 \leq i < j \leq n \tag{4.4.1}$$

The constraints which are equalities at X', are the members of the family

$$\delta_{n-2}(N) \cup \delta_{n-1}(N), \tag{4.4.2}$$

whose cardinality is $\frac{n(n+1)}{2}$. Let B be the submatrix of A^T obtained by choosing the column associated with $\delta_{n-2}(N) \cup \delta_{n-1}(N)$.

Theorem 4.5 :

B is a basis of $\mathbb{DP}(N)$.

Proof : The aim is to show that B is nonsingular. Let $\bar{R}_i$, $\bar{R}_{ij}$ $1 \leq i < j \leq n$ denote the rows of the tableau after some successive row operations. We define $\bar{R}_i$, $\bar{R}_{ij}$ in terms of original rows R_i, R_{ij} as follows,

$$\bar{R}_i = \left[((n-3)R_i - \sum_{j \neq i} R_{ij}) - \frac{1}{n-1}((n-3)\sum_{i=1}^{n} R_i - 2\sum_{1 \leq i < j \leq n} R_{ij})\right](n-1) \tag{4.4.3}$$

$$\bar{R}_{ij} = \frac{2}{n-1}\sum_{t=1}^{n} R_t - \frac{2}{(n-1)(n-2)}\sum_{1 \leq t < k \leq n} R_{ti} - R_i - R_j + R_{ij} \tag{4.4.4}$$

It is not difficult to check that such $\bar{R}_i$, $\bar{R}_{ij}$ transform the matrix B to an identity matrix and hence B is nonsingular.

This completes the proof.

In the tableau discussed earlier, if the vector $\bar{P}$ is written as an additional column then the set of row operations, that converts B to an identity matrix, transforms the column associated with $\bar{P}$ to $B^{-1}\bar{P}$ in the final tableau.

Theorem 4.6 :

X' is optimal if and only if

$$((n-3)P_i - \sum_{j\neq i} P_{ij}) - \frac{1}{n-1}((n-3)\sum_{t=1}^{n} P_t - 2\sum_{1\leq t<k\leq n} P_{tk}) \geq 0 \tag{4.4.5}$$

and

$$\frac{2}{n-1}\sum_{t=1}^{n} P_t - \frac{2}{(n-1)(n-2)}\sum_{1\leq t<k\leq n} P_{tk} - P_i - P_j + P_{ij} \geq 0. \tag{4.4.6}$$

Proof : The $\bar{R}_i$, $\bar{R}_{ij}$, $1 \leq i < j \leq n$ defined in (4.4.3) - (4.4.4) transform the component P_i of $\bar{P}$ to left hand side of (4.4.5) and the component P_{ij} of $\bar{P}$ to left hand side of (4.4.6). Hence the condition (4.4.5) - (4.4.6) is same as $B^{-1}\bar{P} \geq 0$ which is the sufficient condition for optimality.

The 'necessary' part can be derived from the fact that the B obtained from X' is unique for the point X' and this is not the case with arbitrary extreme point of F(N).

This completes the proof.

We next determine the sufficient condition that the extreme point X'' is optimal where X'' is defined as

$$x_i'' = \frac{2}{h+1}\ ,\ x_{ij}'' = \frac{2}{h(h+1)} \quad 1 \le i < j \le n.$$

$$\text{for } 1 \le h \le n-2.$$

The constraints which are equalities at the point X'' are the members of $\delta_h(N) \cup \delta_{h+1}(N)$ whose cardinality is

$$^nC_h + {}^nC_{h+1} \ge \frac{n(n+1)}{2}\ .$$

'e choose the following $\frac{n(n+1)}{2}$ subsets from $\delta_h(N) \cup \delta_{h+1}(N)$ nd form a basis of DP(N).

The subsets are chosen as follows

$\{1,2,\ldots h\} \cup \{j\} \quad \forall\ j \in \{h+1,\ldots,n\}$

$\{1,2,\ldots,h,h+1,h+2\} \setminus \{i\}\ \forall\ i \in \{1,2,\ldots,h\}$

d $\quad \{1,2,\ldots,h-2\} \cup \{i,j\} \quad i < j,\ i,j \in \{h-1,h,\ldots,n\}$

$\{1,2,\ldots,h-2\} \cup \{h-1,h\} \cup \{k\} \setminus \{t\}$

$$\forall\ t \in \{1,2,\ldots,h-2\}$$

$$k \in \{h+1,\ldots,n\}$$

$\{1,2,\ldots,h-2\} \setminus \{t\} \cup \{h-1,h+1\}$

$\{1,2,\ldots,h-2\} \setminus \{t\} \cup \{h,h+1\}$

$$\forall\ t \in \{1,2,\ldots,h-2\}$$

$\{1,2,\ldots,h-2\} \cup \{h-1,h,h+1,h+2\} \setminus \{i,j\}$

$$\forall\ i < j,\ i,j \in \{1,2,\ldots,h-2\}.$$

Let us denote by $\alpha(N)$, the family of above $\frac{n(n+1)}{2}$ subsets. In the following theorem we determine condition that $\alpha(N)$ yields a feasible basis of DP(N). That is to say that if B is the submatrix of A^T chosen with respect to elements of $\alpha(N)$, then B is nonsingular and $B^{-1}\bar{P} \geq 0$. Thus such condition gives the sufficient condition that X'' is optimal point of IP(N). The value of the objective function at X'' is

$$\frac{2}{h+1} \sum_{i=1}^{n} P_i - \frac{2}{h(h+1)} \sum_{1 \leq i < j \leq n} P_{ij} = z' \text{ (say)}$$

Let

$$P_i' = P_i - \frac{1}{h-1} \sum_{j \neq i} P_{ij}$$

$$P_i'' = P_i' - \frac{1}{h+1} \sum_{j=1}^{n} P_j' \qquad \forall\, i = 1,2,\ldots,n.$$

Theorem 4.7 :

$$r_2(\alpha(N)) = |\alpha(N)| = \frac{n(n+1)}{2}$$

and $\alpha(N)$ yields a feasible dual basis if P_i, P_{ij}, $1 \leq i < j \leq n$ satisfy the following set of inequalities.

$$P_i'' \geq 0 \qquad i = 1,2,\ldots,h$$

$$P_{h+1}' - \sum_{i=1}^{h} P_i'' \geq 0$$

$$P_{h+2}' - \sum_{i=1}^{h} P_i'' \geq 0$$

$$P_i' \leq 0, \quad i = h+2,\ldots,n$$

$$P_{ij} - P_i - P_j + z' \geq 0, i = 1,2,\ldots,h-2, \; j = 2,3,\ldots,h, \; i < j$$

$$\sum_{j=1}^{n} P_{ij} - P_{ih-1} - P_{ih} - P_{ih+1} - P_i + P_{h+1} + z' - \sum_{j=h+1}^{n} P_j \geq 0, \; i = 1,2,\ldots,h-2$$

$$\sum_{j=1}^{n} P_{ij} - P_{ih-1} - P_{ih} - P_{ih+2} - P_i + P_{h+2} + z' - \sum_{j=h+1}^{n} P_j \geq 0, i = 1,2,\ldots,h-2$$

$$\frac{1}{h-1} P_{h-1j} + P'_j \geq 0, \; j = h+3,\ldots,n$$

$$\frac{1}{h-1} P_{hj} + P'_j \geq 0 \quad , \; j = h+3,\ldots,n$$

$$P_{h-1h} - \Big[\frac{2}{h} \sum_{1 \leq i < j \leq n} P_{ij} - \frac{h-1}{h+1} \sum_{i=1}^{n} P'_i - \sum_{h-1 \leq i < j \leq n} P_{ij}$$

$$- \sum_{i=1}^{h-2} (P_{ih-1} + P_{ih} + P_i) + (h-2)(P_{h-1} + P_{h+1}) \Big] \geq 0$$

$$P_{st} - \Big[\frac{2}{h} \sum_{1 \leq i < j \leq n} P_{ij} + \frac{h-1}{h+1} \sum_{i=1}^{n} P'_i - \sum_{i=1}^{h-2} (P_{is} + P_{it} + P_i)$$

$$-(P_s + P_t) - \sum_{h-1 \leq i < j \leq n} P_{ij} + \sum_{i=1}^{n} P_{is} + \sum_{i=1}^{n} P_{it} \Big] \geq 0$$

$$s = h-1, \; h, \quad t = h+1, \; h+2$$

$$P_{h+1h+2} - \Big[\sum_{h-2 \leq i < j \leq n} P_{ij} + \sum_{i=1}^{h-2} (P_{ih-1} + P_{ih}) + P_{h-1} + P_h$$

$$- \sum_{i=1}^{n} (P_{ih-1} + P_{ih}) - z' \Big] \geq 0.$$

<u>Proof</u> : The sequence of row operation, that transforms tableau containing B and $\bar{P}$ to I and $B^{-1}\bar{P}$ respectively, is given by

$$\bar{R}_i = (h-1) \left\{ R_i - \frac{1}{h-1} \sum_{j \neq i} R_{ij} - \frac{1}{h+1} \sum_{j=1}^{n} R_j + \frac{2}{(h+1)(h-1)} \sum_{1 \leq i < j \leq n} R_{ij} \right\}$$

$$\bar{R}_i = (h-1)\left\{-R_i + \frac{1}{h-1}\sum_{j\neq i} R_{ij} - \sum_{t=1}^{h}(R_t - \frac{1}{h-1}\sum_{j\neq t} R_{jt}) + \frac{h}{h+1}\sum_{j=1}^{n} R_j - \frac{2h}{(h+1)(h-1)}\sum_{1\leq i<j\leq n} R_{ij}\right\}$$

$$\bar{R}_i = (h-1)\left\{-R_i + \frac{1}{h-1}\sum_{i\neq j} R_{ij}\right\}$$

$$\bar{R}_{ij} = R_{ij}-R_i-R_j+\frac{2}{h+1}\sum_{t=1}^{n} R_t-\frac{2}{h(h+1)}\sum_{1\leq t<k\leq n} R_{tk}$$

$$\bar{R}_{ih+1} = \sum_{t=k+3}^{n} R_{it}-R_{ih-1}-R_{ih}-R_{ih+1}-R_i+R_{h+1} - \sum_{j=h+1}^{n} R_j + \frac{2}{h+1}\sum_{j=1}^{n} R_j - \frac{2}{h(h+1)}\sum_{1\leq j\leq k\leq n} R_{jk}$$

$$\bar{R}_{ij} = R_i - R_{ij} \quad i = 1,2,\ldots,h-2 \quad , \ j = h+3,\ldots,n$$

$$\bar{R}_{h-1j} = (h-1)R_j - \sum_{i\neq j} R_{ij} + R_{h-1j}, \quad j = h+3,\ldots,n$$

$$\bar{R}_{hj} = (h-1)\, R_j - \sum_{i\neq j} R_{ij} + R_{hj}, \quad j = h+3,\ldots,n$$

$$\bar{R}_{h+1j} = R_{h+1j}, \quad j = h+3,\ldots,n$$

$$\bar{R}_{h+2j} = R_{h+2j}, \quad j = h+3,\ldots,n$$

$$\bar{R}_{ij} = R_{ij}, \quad i < j, \ i,j \ h+3,\ldots,n$$

$$\bar{R}_{h-1h} = R_{h-1h}-\left[\frac{2}{h}\sum_{1\leq i<j\leq n} R_{ij}+\frac{2}{h+1}\sum_{1\leq i<j\leq n} R_{ij} - \frac{h-1}{h+1}\sum_{i=1}^{n} R_i-\sum_{h-1\leq j<k\leq n} R_{jk}-\sum_{i=1}^{h-2}(R_{ih-1}+R_{ih}+R_i)+(h-2)(R_{h-1}+R_h)\right]$$

$$\bar{R}_{st} = R_{st} - \left[\frac{2}{h(h+1)} \sum_{1 \leq i < j \leq n} R_{ij} - \sum_{r+1 \leq i < j \leq n} R_{ij} \right.$$

$$+ \sum_{i=1}^{n} (R_{is}+R_{it}) - \sum_{i=1}^{h-2} (R_{is} + R_{it} + R_{i}) - R_{s} - R_{t}$$

$$\left. + \frac{h-1}{h+1} \sum_{i=1}^{n} R_{i} \right], \quad s = h-1,\ h, \quad t = h+1,\ h+2$$

$$\bar{R}_{h+1h+2} = R_{h+1h+2} - \left[\frac{2}{h(h+1)} \sum_{1 \leq i < j \leq n} R_{ij} + \sum_{r+1 \leq i < j \leq n} R_{ij} \right.$$

$$\left. + \sum_{i=1}^{h-2} (R_{ih-1}+R_{ih})+R_{h-1}+R_{h} - \sum_{i=1}^{n} (R_{ih-1}+R_{ih}) - \sum_{i=1}^{n} R_{i} \right]$$

It is merely a routine work to check that such transformations make the matrix B to an identity matrix and $\bar{P}$ to $B^{-1}\bar{P}$. Thus if $B^{-1}\bar{P} \geq 0$ then X'' is optimal. But since the choice of $\alpha(N)$ is not unique for the given point X'', such condition is not unique, and hence the condition given is not necessary but sufficient.

This completes the proof.

As discussed above, one can conceive of several other special cases for which exact lower bounds can be obtained. We, however, do not discuss all these cases here.

Chapter V

LOWER BOUND OF HIGHER DEGREE

5.1 *Introduction*

So far we have restricted our scope of discussion to Bonferroni lower bound of degree two. In this concluding chapter, we give extensions of some results for the cases of lower bound of higher degree. In section 5.2, we generalise the class V^* for the case of lower bound of degree ν. Section 5.3 deals with a generalisation of the results obtained in chapter III. Since the concepts used in this chapter, are nothing but the generalization of the results given in the earlier chapters, here we do not discuss these results elaborately.

5.2 *Bonferroni lower bound of degree* ν

The Bonferroni lower bound problem of degree ν can be formulated as a linear program as follows;

To maximize,

$$LP_\nu(N): \sum_{i_1=1}^{n} x_{i_1} P_{i_1} - \sum_{1 \le i_1 < i_2 \le n} x_{i_1 i_2} P_{i_1 i_2} + \cdots$$

$$\cdots + (-1)^{\nu-1} \sum_{1 \le i_1 < i_2 < \ldots < i_\nu \le n} x_{i_1 i_2 \ldots i_\nu} P_{i_1 i_2 \ldots i_\nu} \tag{5.2.1}$$

subject to $\sum\limits_{i_1 \in J} x_{i_1} - \sum\limits_{\substack{i_1 < i_2 \\ i_1, i_2 \in J}} x_{i_1 i_2} + \cdots$

$$\cdots + (-1)^{\nu-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_\nu \leq n} x_{i_1 i_2 \cdots i_\nu}$$

$$\leq 1 \qquad (5.2.2)$$

for all $J \subseteq N$.

The above linear program $LP_\nu(N)$ is of $\sum\limits_{i=1}^{\nu} {}^nC_i (= m_\nu$, say) variables with 2^n-1 constraints. Let $F_\nu(N)$ denote the set of feasible region of $LP_\nu(N)$. As in the previous case, we define a point $X \in R^{m_\nu}$ with components $x_{i_1}, x_{i_1 i_2}, \ldots, x_{i_1 i_2 \cdots i_\nu}$ as

$$X = (x_1, x_2, \ldots, x_n, x_{12}, \ldots, x_{n-1n}, \ldots, x_{123\ldots\nu}, \ldots, x_{(n-\nu+1)\ldots n})$$

We recall that the class V^* introduced in chapter II defines a subset of feasible points determined by integers $h \geq 1$, a_i, $i \in N$. Such a class of points can also be defined for the general case as follows.

Let us consider the polynomial

$$(h - \sum_{i=1}^{n} a_i)(h - \sum_{i=1}^{n} a_i + 1) \cdots (h - \sum_{i=1}^{n} a_i + \nu - 1) \qquad (5.2.3)$$

where the variables are h, a_i, $i \in N$. This polynomial contains terms like $t\, h^p\, a_{i_1}^{q_1}$, $t\, h^p\, a_{i_1}^{q_1} a_{i_2}^{q_2}, \ldots, t\, h^p\, a_{i_1}^{q_1} a_{i_2}^{q_2} \cdots a_{i_\nu}^{q_\nu}$,

where, t, p_i, q_i are constants.

We define the class $V^{*\nu}$, like V^* as, for a given integer vector $(a,h) \in R^{n+1}$, $h \geq 1$

$$x_{i_1} = \frac{1}{h(h+1)(h+2)\ldots(h+\nu-1)} \Big[\text{sum of the terms of the type } t\, h^p\, a_{i_1}^{q_1} \text{ in } (5.2.3)\Big], \quad q_1 > 0$$

$$x_{i_1 i_2} = \frac{1}{h(h+1)(h+2)\ldots(h+\nu-1)} \Big[\text{sum of the terms of the type } t\, h^p\, a_{i_1}^{q_1}\, a_{i_2}^{q_2} \text{ in } (5.2.3)\Big], \quad q_1 > 0,\ q_2 > 0$$

and similarly

$$x_{i_1 i_2 \ldots i_\nu} = \frac{1}{h(h+1)(h+2)\ldots(h+\nu-1)} \Big[\text{sum of the terms of the type } t\, h^p\, a_{i_1}^{q_1}\, a_{i_2}^{q_2} \ldots a_{i_\nu}^{q_\nu} \text{ in } (5.2.3)\Big], \quad q_i > 0,\ 1 \leq i \leq \nu.$$

In theorem 5.1, we show that the class $V^{*\nu}$ is a subset of the feasible region of $LP_\nu(N)$.

<u>Theorem 5.1</u> :

Any $X \in V^{*\nu} \subseteq R^{m_\nu}$ is feasible to the set of constraints (5.2.2).

<u>Proof</u> : Substituting the value of $X \in R^{m_\nu}$ in terms of (a,h) in the constraint associated with any subset $J \subseteq N$, we get

$$\left[\frac{(h - \sum_{i \in J} a_i)(h - \sum_{i \in J} a_i + 1) \ldots (h - \sum_{i \in J} a_i + \nu - 1)}{h(h+1)(h+2) \ldots (h+\nu-1)} \right] \geq 0. \tag{5.2.5}$$

It is clear that for any integers $h \geq 1$, a_i, $i \in N$, the inequality (5.2.5) is true. Hence the constraint J is satisfied by $X \in V^{*\nu}$.

This completes the proof.

A special subset of $V^{*\nu}$ can be defined as follows, For a given $S \subseteq N$, for integer h, $1 \leq h \leq s-\nu$,

$$x_{i_1} = \frac{\nu}{(h+\nu-1)} \quad , \quad \forall\, i_1 \in S$$

$$x_{i_1 i_2} = \frac{\nu(\nu-1)}{(h+\nu-1)(h+\nu-2)} \quad , \quad i_1 < i_2,\ i_1, i_2 \in S,$$

and similarly,

$$x_{i_1 i_2 \ldots i_\nu} = \frac{\nu(\nu-1)\ \ldots\ 2\ 1}{h(h+1)\ \ldots\ (h+\nu-1)} \quad , \quad \forall\, i_1 < i_2 < \ldots < i_\nu \quad i_1, i_2 \ldots i_\nu \in S.$$

and $x_{i_1} = x_{i_1 i_2} = \ldots = x_{i_1 i_2 \ldots i_\nu} = 0$, otherwise.

It is easy to see that for $\nu = 2$ such a kind of point give rise to class V_2 defined in (1.5.2). This special class is useful to get a lower bound of degree ν . [29], [47].

Substituting the value of X in terms of (a,h) in (5.2.1). We see that the computation of Bonferroni lower bound of degree ν can be viewed as a polynomial of degree ν of integer variables. At present we do not discuss the computational aspect of solving such problems.

5.3 *Extreme points of $F_\nu(N)$*

In this section we give some general results regarding the extreme points of $F_\nu(N)$.

Any constraint of (5.2.2) can be written as

$$d(J)^T X \leq 1$$

where $d(J) \in R^{m_\nu}$ is defined as

$$d(J) = (d_1, d_2, \ldots, d_n, d_{12}, \ldots, d_{n-1n}, \ldots, d_{(n-\nu+1)\ldots n})$$

such that

$$\begin{aligned} d_{i_1} &= 1 \quad \text{if } i_1 \in J \\ &= 0 \quad \text{if } i_1 \notin J \end{aligned}$$

and

$$d_{i_1 i_2 \ldots i_j} = (-1)^{j-1}\, d_{i_1} d_{i_2} \ldots d_{i_j}, \quad 1 \leq j \leq \nu .$$

For such given vector $d(J) \in R^{m_\nu}$ we define a vector $d_i(J) \in R^{m_i}$ as the first $m_i (= \sum_{j=1}^{i} {}^nC_j)$ components of $d(J)$. Like the previous case, we introduce the ranks of different orders. For a family of subsets of N, $\alpha(N)$, the rank of order i, $1 \leq i \leq \nu$ is defined as the maximum number of linearly independent vectors in the set $\{d_i(J) \mid J \in \alpha(N)\}$, and this rank is denoted by $r_i(\alpha(N))$.

In theorem 5.2, we give a characterisation of extreme points of first kind of $F_\nu(N)$.

Theorem 5.2 :

For a given subset $S \subseteq N$, let integers $h \geq 1$, $a_i, i \in N\setminus S$ be chosen such that

(i) $1 \leq h \leq s-\nu$

(ii) $\exists$ a collection $\alpha(N\setminus S)$ of $N\setminus S$ such that

$$|\alpha(N\setminus S)| = \sum_{i=1}^{\nu} {}^{n-s}C_i .$$

$$r_\nu(\alpha(N\setminus S)) = \sum_{i=1}^{\nu} {}^{n-s}C_i .$$

(iii) $0 \leq \theta(I) \leq s-\nu+1$, $\forall\ I \in \alpha(N\setminus S)$ where $\theta(I) = h - \sum_{i \in I} a_i$.

Then the point X of $V^{*\nu}$ defined by $X = (a,h)$ such that $a_i = 1$, $i \in S$ is an extreme point of $F_\nu(N)$.

The proof of the theorem is omitted here because of the fact that the theorem can be proved following the same arguments as used in theorem 3.3.

Remark : Here it is to be noted that like any point in Ext (1), any extreme point of first kind of $F_\nu(N)$ satisfies ν consecutive blocks of any particular column of $R(S)$, which is independent of degree ν.

Using the concept of rank of order i, we proceed to generalise the idea of extreme points of different kind.

Let for a given $S \subseteq N$, and given integers a_i, h, $i \in N\setminus S$ the subsets of $N\setminus S$ are partitioned as π, π_1, π_2 as given in

section 3.4. Let us assume that $r_\nu(\pi) < \sum_{i=1}^{\nu} {}^{n-s}C_i$ and there exists $\beta(N\setminus S) \subseteq \pi$ such that

$$r_{\nu-1}(\beta(N\setminus S)) = |\beta(N\setminus S)| = \sum_{i=1}^{\nu-1} {}^{n-s}C_i .$$

Then the point $(a,h) \in V^{*\nu}$ with $a_i = 1$, $i \in S$ can be extended to generate an extreme point which is called an extreme points of second kind. Such concept is useful to define ν different kinds of extreme points for the $LP_\nu(N)$.

We, however, do not pursue it any further.

REFERENCES

1. Agarwal, S.C., "On mixed integer quadratic program", N.R.L.Q. 21(1974), pp. 287-298.

2. Balas, E., "Duality in discrete programming : quadratic case", Management Science, 16 (1969), pp. 14-32.

3. Bashurov, V.V., "The solution of an integer quadratic programming problem", U.S.S.R. Computational Maths. and Mathematical Physics, 6(1966), pp. 270-273.

4. Berman, G., "A branch and bound method for the maximization of Pseudo-Boolean function", Tech. Report, Faculty of Mathematics, University of Waterloo.

5. Bonferroni, C.E., "Teoria stastiticaclassi e'calculodella probability", Pubbl. R. Ist. Super Sci. Econ. Comm; Firenze, 8(1936), pp. 1-62.

6. Boot, J.C.G. and Their, H., " A procedure for integer maximization of a definite quadratic function," in G. Kreweras and G. Morlat (Eds)" Actes dela 3 IEME Conf. Internationale De Recherche operationalle." (1964), pp. 667-682.

7. Boole, G., "<u>Studies in logic and probabilities</u>", Walts and Co., 1952.

8. Chung, K.L. and Erdos, P., " On the application of Borel-Cantelli lemma", Trans. Am. Math. Soc., 72 (1952), pp. 179-186.

9. Dinkleback, W., "On convex integer programming," in A. Prekopa (Ed.) "Colloquim on application of mathematics to economics," Budapest, 1963, pp. 75-78.

10. Dowson, D.A. and Sankoff, D., " An inequality for probabilities," Proc. Amer. Math. Soc., 18 (1967), pp. 504-507.

11. Frechet, M., 'Les probabilities associees a'unsysteme d' evenments compatables et dependents," Actualities Scientifiques et Industrielles, Nos. 859-942, (1943).

12. Gaiha, P. and Gupta, S.K., "Adjacent vertices on a permutohedron," SIAM Jr. Appl. Maths., 32 (1977), pp. 323-327.

13. Galambos, J., "Bonferroni inequalities," The Ann. of Prob., 5(1977), pp. 577-581.

14. Galambos, J., "The assymptotic theory of extreme order statistics," John Wiley, N.Y., 1978.

15. Galambos, J., "Methods of proving Bonferroni type inequalities," Jr. London Math. Soc., 2(1975), pp. 561-564.

16. Galambos, J. and Mucci, R., "Inequalities for combinations of binomial moments," private communication.

17. Gallot, S., "A bound for the maximum of a number of random variables," Jr. of Appl. Prob., 3(1966), pp. 556-558.

18. Hadley, G., "Linear Programming," Addison-Wesley, World student series, 1969.

19. Hailperin, T., "Best possible inequalities for the probabilities of a logical function of events", Amer. Math. monthly, 72 (1965), pp. 343-359.

20. Hailperin, T., "Boole's logic and probability," Studies in logic and foundation of mathematics, 85 (1976).

21. Hammer, P.L., "Boolean procedure for bivalent programming," in P.L. Hammer and G. Zountedijk (Eds.), Mathematical Programming and application, North-holland/America.

22. Hammer, P.L., "BBB method I, maximization of a Pseudo-Boolean function," Publ. Centre De Recherche Mathematiqus, Universite De Montreal, No. 220 (1972).

23. Hammer, P.L. and Hansen, P., "Quadratic 0-1 programming," Core discussion paper 7219, April, 1972.

24. Hammer, P.L. and Peled, U.N., "On the maximization of a Pseudo-Boolean function," JACM, 19 (1972), pp. 165-282.

25. Hammer, P.L. and Rosenberg, I.G., "Equivalent form of 0-1 programs," in Zaremba (Eds.), In application of number theory to numerical analysis, Academic Press, New York and London, 1972, pp. 453-463.

26. Hammer, P.L. and Rubin, A., "Some remarks on quadratic programming with 0-1 variables," Revue Francaise D' Informatique, Et DE Recherche Operationelle, 4(1970), V-3, pp. 67-69.

27. Hammer, P.L. and Rudeanu, S., " Boolean methods in operation research and related areas," Springer-Verlag, Berlin-Heidelberg-New York, 1968.

28. Hansen, P., "Quadratic 0-1 programming by implicit enumeriation," in F.A. Lootsma (Ed.), Numerical methods for nonlinear optimization, Academic Press, 1972, London.

29. Kounias, E.G., "Bounds for the probability of unions with applications," The Annals of Math. Stats. 39 (1968), pp. 2154-2158.

30. Kounias, S. and Domianou, Ch., " On an inequality in probability," Dis. Maths., 22 (1978), pp. 85-89.

31. Kounias, S. and Marin, J., "Best linear Bonferroni bounds," SIAM Jr. of Appl. Maths., 30 (1976), pp. 307-323.

32. Kuenzi, H.P. and Oettli, W., "Integer quadratic programming," in R.L. Graves, and P. Wolfe (Eds.), Recent advances in Mathematical Programming (303-308), New York, McGraw-Hill, 1963.

33. Kwerel, S.M., "Information retrieval for media planning," Management Science, 15 (1968), pp. 137-60.

34. Kwerel, S.M., "Most stringent bounds on aggregate probabilities of partially specified dependent probability system," Jr. Amer. Stats. Assoc., 70 (1975), pp. 474-479.

35. Laughnn, D.J., "Quadratic binary programming with application to capital budgeting problem," O.R., 18 (1970), pp. 459-461.

36. Laughnn, D.J., " Capital expenditure programming and some alternative approaches to risk," Management Science, 17 (1971), No. 5, pp. 320-336.

37. Mallows, C.L., "An inequality involving multinomial probabilities," Biometrica, 55 (1968), pp. 422-424.

38. McCallum, C.J., "An algorithm for certain quadratic integers problem," Working paper, Bell Lab., New Jersey, 1975.

39. Murty, K.G., " Linear and combinatorial programming," John Wiley interscience, 1976.

40. Olkin, I. and Sobel, M., "Integral expression for tail probabilities of the multinomial and negative multinomial distribution," Biometrica, 52 (1965), pp. 167-179.

41. Renyi, A., "Foundation of probability," Holden-day-Sanfrancisco, Calif, 1970.

42. Rosenberg, I.G., "0-1 optimization and non-linear programming Publication CRM-165, Revue Francaise d' Informatique et de Recherche operationelle 2(1972), pp. 95-97.

43. Rosenberg, I.G., " Minimization of a Pseudo-Boolean function by binary development," Discrete Mathematics, 7 (1974), pp. 151-165.

44. Rosenberg, I.G., " Reduction of bivalent maximization to the quadratic case," Cahiers Centre Etode Recherche Oper., 17 (1975), pp. 71-79.

45. Roy, B., (Ed) "Combinatorial programming, methods and applications," D. Reidel publishing company, 1974.

46. Saaty, T.L., " On nonlinear optimization in integers," N.R.L.Q., 15 (1968), pp. 1-22.

47. Sobel, M. and Uppuluri, V.R.R., " On Bonferroni type inequalities of the same degreefor the probability of unions and intersections," The Anals of Maths. Stats. 43 (1972), 5, pp. 1549-1558.

48. Stoer, J. and Witzgall, C., "Convexity and optimization infinite dimension, I," Springer-Verlag, New York, 1970.

49. Swick, D.A., "Harmonic relationships among the random variables," SIAM Jr. of Appl. Math. 33, (1977), pp. 490-497.

50. Walkup, D.W. and Wets, R., J.-B., "Lifting projection of convex polyhedra," Pacific Jr. of Math., 28 (1969), pp. 465-475.

51. Wets, R., J.-B. and Witzgall, C., "Towards an algebraic characterization of convex polyhedron cones," Numerische Mathematik, 12 (1968), pp. 134-128.

52. Whittle, P., "Sur la distribution du maximum d' un polynome trigonometrique a coefficients a leatories," Le cacul des Probabilities et sel Applications, CNRS, 87 (1959), pp. 173-184.

53. Yoshida, Y., Inagaki, Y. and Fukumura, T., " Algorithm for Pseudo-Boolean program based on branch and bound method," Jr. Inst. Elect. and Comm. Eng., Japan, 50 (1967), pp. 277-285.

Here we give Q matrices for the problems discussed in table 2.3.

Problem 1

$$= \begin{bmatrix} 0.495 & 0.244 & 0.244 & 0.244 & 0.118 & 0.322 \\ 0.244 & 0.495 & 0.244 & 0.244 & 0.118 & 0.322 \\ 0.244 & 0.244 & 0.495 & 0.244 & 0.118 & 0.322 \\ 0.244 & 0.244 & 0.244 & 0.495 & 0.118 & 0.322 \\ 0.118 & 0.118 & 0.118 & 0.118 & 0.431 & 0.284 \\ 0.322 & 0.322 & 0.322 & 0.322 & 0.284 & 0.591 \end{bmatrix}$$

Problem 2

$$Q = \begin{bmatrix} 0.225 & 0.070 & 0.080 & 0.090 & 0.040 & 0.020 \\ 0.070 & 0.240 & 0.095 & 0.105 & 0.035 & 0.015 \\ 0.080 & 0.095 & 0.245 & 0.115 & 0.030 & 0.010 \\ 0.090 & 0.105 & 0.115 & 0.250 & 0.025 & 0.005 \\ 0.040 & 0.035 & 0.030 & 0.025 & 0.160 & 0.010 \\ 0.020 & 0.015 & 0.010 & 0.005 & 0.010 & 0.075 \end{bmatrix}$$

Problem 3

$$Q = \begin{bmatrix} 0.385 & 0.210 & 0.190 & 0.210 & 0.215 & 0.040 \\ 0.210 & 0.390 & 0.180 & 0.210 & 0.220 & 0.045 \\ 0.190 & 0.180 & 0.360 & 0.215 & 0.215 & 0.030 \\ 0.210 & 0.210 & 0.215 & 0.390 & 0.220 & 0.035 \\ 0.215 & 0.220 & 0.215 & 0.220 & 0.385 & 0.025 \\ 0.040 & 0.045 & 0.030 & 0.035 & 0.025 & 0.200 \end{bmatrix}$$

Problem 4

$$Q = \begin{bmatrix} 0.563 & 0.393 & 0.162 & 0.264 & 0.340 & 0.138 \\ 0.393 & 0.563 & 0.268 & 0.221 & 0.231 & 0.079 \\ 0.160 & 0.268 & 0.384 & 0.107 & 0.179 & 0.161 \\ 0.264 & 0.221 & 0.107 & 0.437 & 0.255 & 0.079 \\ 0.340 & 0.231 & 0.179 & 0.255 & 0.522 & 0.109 \\ 0.138 & 0.070 & 0.161 & 0.079 & 0.109 & 0.220 \end{bmatrix}$$

A 65922

Date Slip

This book is to be returned on the date last stamped.

....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....

CD 6.72.9